Int. J. Multiphase Flow Vol. 8, No. 4, pp. 361–392, 1982
Printed in Great Britain.

0301-9322/82/040361-32$03.00/0
Pergamon/Elsevier

TWO PHASE STOKES FLOWS DISTORTED BY A SPHERE STRADDLING THE INTERFACE

A. M. J. DAVIS
Department of Mathematics, University College, Gower Street, London WCIE 6BT, England

(*Received* 18 *May* 1981; *in revised form* 2 *November* 1981)

Abstract—The accuracy of the approximate expression given by Ranger (1978) for the force exerted by a two-phase Stokes flow on a sphere which symmetrically straddles the interface, is examined analytically and shown to depend critically on the ability of the surface tension or gravity forces to resist deformation of the interface.

1. INTRODUCTION

The motion of a particle moving in the presence of a free fluid–fluid interface is of considerable importance and interest in chemical engineering science. The general motion of an arbitrary particle in the presence of such a free interface is exceedingly complex and several simplifying assumptions are necessary to render analytical solutions possible. One such is to assume that the surface tension forces are able to resist deformations of the interface and hence to replace the normal stress condition by one specifying the shape of the interface. This is effectively used for droplets but in the current context becomes less acceptable as the body is placed closer to the interface. The exception considered by Ranger (1978) is a disk lying in a planar interface and moving parallel to it, for in this case the normal stresses are identically zero and no distortion of the interface can occur.

Compared with a sphere straddling the interface, the choice of the disk circumvents the major difficulty arising from the occurrence of a normal stress discontinuity in a region containing a stagnation point. In the same paper, Ranger suggests that an axisymmetric Stokes flow past the sphere can provide a useful approximation to the two phase flow but does not consider in detail its accuracy. This paper aims to rectify this omission by considering perturbations with respect to the fractional viscosity difference defined in the next section. In this way it is expected to obtain information concerning how the original Stokes flow and planar interface are changed by increasing from zero the viscosity difference. A density discontinuity is also introduced into the two basic flows, stagnation and streaming, which are considered here. For the symmetrically placed sphere, exact solutions of the equations of motion are sought in terms of toroidal coordinates for the first order perturbation fields when the surface tension and/or gravity forces are strong or weak. The stabilizing effect of these folces is amply demonstrated and the dominant presence of at least one is found to be necessary for Ranger's force approximation to be accurate. It is shown that no exact solution of the assumed form exists for the streaming motion in the limit of zero surface tension and identical densities, a plausible result in the absence of any "restoring" mechanism capable of restraining the "deforming" mechanism of the viscous stresses.

2. FORMULATION OF THE PROBLEM

On choosing Cartesian coordinates $0xyz$, with the z-axis directed vertically upwards, let $\mathbf{v}_0(x, y, z)$ be a Stokes flow of a uniform fluid of viscosity μ such that the plane $z = 0$ is stress-free. Since the gravitational acceleration $-g\hat{z}$ can be accounted for by suitably modifying the pressure $p(x, y, z)$, the above mentioned flow field $\mathbf{v}_0$ is unchanged by the introduction of a density discontinuity $\Delta(>0)$ at the plane $z = 0$. Also, this flow field is further unaffected when the viscosity is changed to $\mu_1 = \mu(1+\lambda)$ in $z > 0$ and $\mu_2 = \mu(1-\lambda)$ in $z < 0$ (Davis *et al.* 1975), where the parameter

$$\lambda = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \qquad [1]$$

is evidently such that $|\lambda| < 1$. Note here that if values of less than $\frac{1}{3}$ may be regarded as small, the corresponding range of the viscosity ratio is $\frac{1}{2} < \mu_1/\mu_2 < 2$.

Now suppose that this two-fluid flow is distorted by the introduction of a fixed rigid sphere which may be placed anywhere, though positions near or straddling the plane $z = 0$ provide the greatest interest. The resulting flow is such that the interface is no longer at $z = 0$ and its detailed calculation is of considerable complexity. Some indications of the effects of introducing the viscosity discontinuity can be obtained by constructing the first order solution for each choice of $\mathbf{v}_0$ and position of the sphere, in a perturbation scheme based on the $\lambda = 0$ but $\Delta \neq 0$ case.

Suppose that the introduction of the sphere into the flow $\mathbf{v}_0$ of two fluids of uniform viscosity μ but different densities results in a flow $\mathbf{q}_0(x, y, z)$ in which $z = z_0(x, y)$ is the stream surface such that $z_0 \to 0$ as $x^2 + y^2 \to \infty$. The velocity field $\mathbf{q}_0$ satisfies the no-slip condition at fixed boundaries and is asymptotic to $\mathbf{v}_0$ at large distances. Writing the pressure $p(x, y, z)$ in the form

$$p = \mu P \qquad [2]$$

throughout, the creeping motion equations satisfied by $\mathbf{q}_0$ and the corresponding P_0 are

$$\operatorname{div} \mathbf{q}_0 = 0, \quad \operatorname{grad} P_0 = \nabla^2 \mathbf{q}_0. \qquad [3]$$

The possibility that the convective terms become significant at large enough distances is ignored here because only the dominant features of the flow near the sphere are sought.

The first order perturbation scheme involves writing the velocity and pressure fields in the form

$$\mathbf{q}^{(j)} = \mathbf{q}_0 + \lambda \mathbf{q}_1^{(j)}, \quad P^{(j)} = P_0 + \lambda P_1^{(j)} \quad (j = 1, 2) \qquad [4]$$

for each fluid of viscosity μ_j, whilst the interface is given by

$$z = z_0(x, y) + \lambda f(x, y). \qquad [5]$$

Then the boundary conditions at the interface, including surface tension forces, are linearised about $z = z_0$, which procedure requires the assumption that the first derivatives of f are also of order unity. The condition that the plane $z = 0$ be stress-free in the flow $\mathbf{v}_0$ ensures that

$$f(x, y) \to \text{constant as } x^2 + y^2 \to \infty.$$

The elementary choices of $\mathbf{v}_0$ considered in this paper are a double-sided axisymmetric stagnation flow and a uniform flow. For these, the length scale of the perturbed flows is determined by the radius of the sphere, which for convenience is taken to be unity. It is also convenient to always choose the origin so that the centre of the sphere lies on the z-axis. Then, on introducing cylindrical polar coordinates (ρ, ϕ, z), the above flows $\mathbf{v}_0$ take the forms

(i) $$\mathbf{v}_0 = Vz\hat{z} - \tfrac{1}{2}V\rho\hat{\rho},$$

(ii) $$\mathbf{v}_0 = V\hat{x},$$

(iii) $$\mathbf{v}_0 = Vz\hat{z} - \tfrac{1}{2}V[(x - x_0)\hat{x} + y\hat{y}] = Vz\hat{z} - \tfrac{1}{2}V\rho\hat{\rho} + \tfrac{1}{2}Vx_0\hat{x}.$$

Case (ii) is the uniform flow, whilst cases (i) and (iii) are respectively the stagnation flow with axis centrally placed at $\rho = 0$ or off-centre at the line $x = x_0$, $y = 0$. In order that the plane $z = 0$ remain stress-free when the viscosity discontinuity is introduced, it is necessary in cases (i) and (iii) that a uniform pressure difference $2(\mu_1 - \mu_2)V$ be maintained between the fluids of viscosities μ_1, μ_2. Evidently case (iii) may be regarded as a superposition of cases (i) and (ii). Also, since the length scale is unity, the constant V is the velocity scale in all three cases.

3. THE SYMMETRICALLY PLACED SPHERE

The position of the sphere, which offers the best chance of constructing a mathematical analysis of the problem stated in the previous section, is that in which its centre is at the origin. Then the flows $(\mathbf{q}_0, P_0)$ are given for the two principal cases by

(i)

$$\mathbf{q}_0 = (Vz\hat{z} - \tfrac{1}{2}V\rho\hat{\rho})\left[1 - \frac{5}{2(\rho^2+z^2)^{3/2}} + \frac{3}{2(\rho^2+z^2)^{5/2}}\right] + \frac{15Vz}{4(\rho^2+z^2)^{5/2}}(\rho^2\hat{z} - z\rho\hat{\rho})\left(1 - \frac{1}{\rho^2+z^2}\right) \qquad [6]$$

$$P_0 = V\left(2 - \frac{5}{(\rho^2+z^2)^{3/2}} + \frac{15\rho^2}{2(\rho^2+z^2)^{5/2}}\right)$$

(ii)

$$\mathbf{q}_0 = V\hat{x}\left[1 - \frac{3}{2(x^2+y^2+z^2)^{1/2}} + \frac{1}{2(x^2+y^2+z^2)^{3/2}}\right] + \frac{3V}{4}\frac{[(y^2+z^2)\hat{x} - x(y\hat{y} + z\hat{z})]}{(x^2+y^2+z^2)^{3/2}}\left(1 - \frac{1}{x^2+y^2+z^2}\right) \qquad [7]$$

$$P_0 = -\frac{3Vx}{2(x^2+y^2+z^2)^{3/2}}.$$

In each case q_{0z} is an odd function of z but q_{0x}, q_{0y} and P_0 are even in z. In terms of spherical polar coordinates (r, θ, ϕ) these expressions may be written:

(i)

$$\mathbf{q}_0 = Vr\hat{r}\left(1 - \frac{5}{2r^3} + \frac{3}{2r^5}\right)P_2(\cos\theta) - \frac{3V}{2}\hat{\theta}\left(r - \frac{1}{r^4}\right)\sin\theta\cos\theta$$

$$P_0 = V\left(2 - \frac{5}{r^3}P_2(\cos\theta)\right).$$

(ii)

$$\mathbf{q}_0 = V\hat{r}\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\sin\theta\cos\phi + V\left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right)(\hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi)$$

$$P_0 = -\frac{3V}{2r^2}\sin\theta\cos\phi,$$

where P_2 denotes a Legendre polynomial. It then readily follows that in the two-fluid flow, the zero order velocity field $\mathbf{q}_0$ exerts on the sphere a force $3\pi(\mu_1 - \mu_2)V\hat{z}$ in case (i) and a force $3\pi(\mu_1 + \mu_2)V\hat{x}$ and torque $\frac{3}{2}\pi(\mu_1 - \mu_2)V\hat{y}$ in case (ii).

Proceeding to consider the first order perturbations introduced by [4] and [5], it may first be noted that since $z = 0$ is a plane of symmetry in the flows $\mathbf{q}_0$ given by [6] and [7] and the direction of the gravitational force cannot be reversed without also changing the sign of Δ, the position of the interface must be an odd function of λ, i.e.

$$z = \lambda f(\rho, \phi) + 0(\lambda^3). \qquad [8]$$

The unit normal to the interface is then given by

$$\hat{n} = \hat{z} - \lambda \frac{\partial f}{\partial \rho}\hat{\rho} - \frac{\lambda}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + 0(\lambda^2). \qquad [9]$$

Linearisation about $z = 0$ enables the values of the velocity and stress components and the pressure to be written in terms of those at the undistrubed interface. Thus, e.g. to order λ,

$$\mathbf{q}(\rho, \phi, \lambda f(\rho,\phi)) = \mathbf{q}(\rho, \phi, 0) + \frac{\partial \mathbf{q}}{\partial z}(\rho, \phi, 0)\lambda f(\rho, \phi). \qquad [10]$$

In this way it becomes possible to apply all the interface conditions at the plane $z = 0$, a known location.

The continuity of the velocity components requires that

$$\mathbf{q}_1^{(1)} = \mathbf{q}_1^{(2)}, \quad \text{at } z = 0, \quad \rho > 1. \qquad [11]$$

Further, since the velocity normal to the interface must be zero in the steady solution considered, the $0(\lambda)$ terms in the equation $\mathbf{q} \cdot \hat{n} = 0$ imply that

$$q_{1z}^{(j)} + f\frac{\partial q_{0z}}{\partial z} - \frac{\partial f}{\partial \rho} q_{0\rho} - \frac{1}{\rho}\frac{\partial f}{\partial \phi} q_{0\phi} = 0 \quad (j = 1.2) \text{ at } z = 0, \quad \rho > 1. \qquad [12]$$

The stresses acting across the interface are, in either fluid,

$$\begin{aligned} &\sigma_{zz}\hat{z} + \sigma_{\rho z}\hat{\rho} + \sigma_{\phi z}\hat{\phi} \\ &- \lambda \frac{\partial f}{\partial \rho}(\sigma_{z\rho}\hat{z} + \sigma_{\rho\rho}\hat{\rho} + \sigma_{\phi}\hat{\phi}) \\ &- \frac{\lambda}{\rho}\frac{\partial f}{\partial \phi}(\sigma_{z\phi}\hat{z} + \sigma_{\rho\phi}\hat{\rho} + \sigma_{\phi\phi}\hat{\phi}), \end{aligned}$$

in standard notation. In the zero order flow, $\sigma_{\rho z}$ and $\sigma_{\phi z}$ vanish at $z = 0$ so the introduction of the viscosity discontinuity into this flow causes only a normal stress discontinuity, namely

$$(\mu_1 - \mu_2)\left[-P_0 + 2\frac{\partial q_{0z}}{\partial z}\right]_{z=0} \hat{z} = 2\mu\lambda \left[-P_0 + 2\frac{\partial q_{0z}}{\partial z}\right]_{z=0} \hat{z}.$$

This is the one feature which prevents the interface from remaining at the plane $z = 0$, the position it takes in the absence of the sphere. On equating the normal stress discontinuity to the surface tension force, the $0(\lambda)$ terms require that

$$\begin{aligned} &-[P_1^{(1)} - P_1^{(2)}]_{z=0} + 2\left[-P_0 + 2\frac{\partial q_{0z}}{\partial z}\right]_{z=0} \\ &+ \frac{\gamma}{\mu}\left(\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2}\right) - \frac{\Delta g}{\mu} f = 0 \ (\rho > 1) \end{aligned} \qquad [13]$$

where γ is the surface tension coefficient and the gravitational term has appeared because the unmodified pressure must be inserted in the normal stress expression. The term $\partial[q_{1z}^{(1)} - q_{1z}^{(2)}]/\partial z$ does not contribute to [13] because div $\mathbf{q}_1^{(j)} = 0$ and the tangential components of velocity are continuous. The stipulation, in section 2, that the flow $\mathbf{v}_0$ be stress-free at $z = 0$, is necessary to ensure that the second term in [13] tends to zero as $\rho \to \infty$. Finally the continuity of tangential stress, to order λ, at the interface, requires that $\sigma_{\rho z}$ and $\sigma_{\phi z}$ be continuous at $z = 0$, to this order, i.e.

$$\frac{\partial}{\partial z}[q_{1\rho}^{(1)} - q_{1\rho}^{(2)}] = 0, \quad \frac{\partial}{\partial z}[q_{1\phi}^{(1)} - q_{1\phi}^{(2)}] = 0 \quad \text{at } z = 0, \quad \rho > 1 \qquad [14]$$

The interface conditions [11]–[14] complete the specification of the first order perturbation fields $\mathbf{q}_1^{(j)}$, $P_1^{(j)}$ ($j = 1, 2$) which must satisfy Stokes equations [3] and have zero velocity on the sphere and at infinity. Of the two conditions involving f, the inhomogeneous one [13] shows that the solution is a function of the parameters $\mu V/\gamma$ and $\mu V/\Delta g$ which determine the relative contribution of the perturbation field, surface tension and gravity to the counterbalancing of the normal stress discontinuity in the zero order flow $\mathbf{q}_0$. The dimensionless parameter $\mu V/\gamma$ is a possible meaning of the term Capillary Number and measures the ratio of viscous and surface tension forces. The parameter $\mu V/\Delta g$ has dimension $[L^2]$ and therefore must be expressed in terms of the radius a of the sphere which is the length unit adopted here. Equivalently $\mu V/\Delta g a^2$ is a dimensionless parameter which measures the ratio of viscous and gravity forces and is the product of the above mentioned Capillary Number and one of several possible definitions of Weber Number. The nomenclature quoted here appears in the extensive listings of the Handbook of Chemistry and Physics (1979).

The three obvious approximations to consider are:

(a) $\mu V/\gamma \ll 1$, $\Delta g/\gamma \ll 1$; dominant surface tension.

Here the first term of [13] is neglected and $f(\rho, \phi)$ is determined directly from the zero order flow. Since then $f = 0(\mu V/\gamma)$, this approximation indicates the accuracy of the commonly made assumption that surface tension forces are large enough to hold the position of the interface against the effects of a discontinuity in normal stress. With f determined, subsequent substitution in [12] yields prescribed values at $z = 0$ of $q_{1z}^{(j)}$ from which $\mathbf{q}_1^{(j)}$ must evidently be of order $\mu V^2/\gamma$.

(b) $\mu V/\Delta g \ll 1$, $\Delta g/\gamma \gg 1$; gravity dominant.

This case is similar to (a) with $f = 0(\mu V/\Delta g)$ and the gravitational forces are large enough to restrict the effects of a normal stress discontinuity on the position of the interface. $\mathbf{q}_1^{(j)}$ is here of order $\mu V^2/\Delta g$.

(c) $\mu V/\gamma \gg 1$, $\mu V/\Delta g \gg 1$; viscosity dominant.

Here the third term of [13] is neglected and only viscosity effects are retained in the calculation. Then $\mathbf{q}_1^{(j)}$ is of order V and is determined independently of f which is subsequently found from [12] as the solution of a differential equation.

On substitution of [6] and [7], the interface conditions [12] and [13] involving f become, in the two cases, illustrated by figure 1:

(i)

$$q_{1z}^{(j)} + \frac{V}{2\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\left(\rho^2 - \frac{5}{2\rho} + \frac{3}{2\rho^2}\right)f\right] = 0 \quad (j = 1, 2) \quad \text{at } z = 0, \rho > 1 \qquad [15]$$

$$[P_1^{(1)} - P_1^{(2)}]_{z=0} = -\frac{9V}{\rho^5} + \frac{\gamma}{\mu\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}f}{\mathrm{d}\rho}\right) - \frac{\Delta g}{\mu}f \quad (\rho > 1) \qquad [16]$$

since, evidently $f = f(\rho)$.

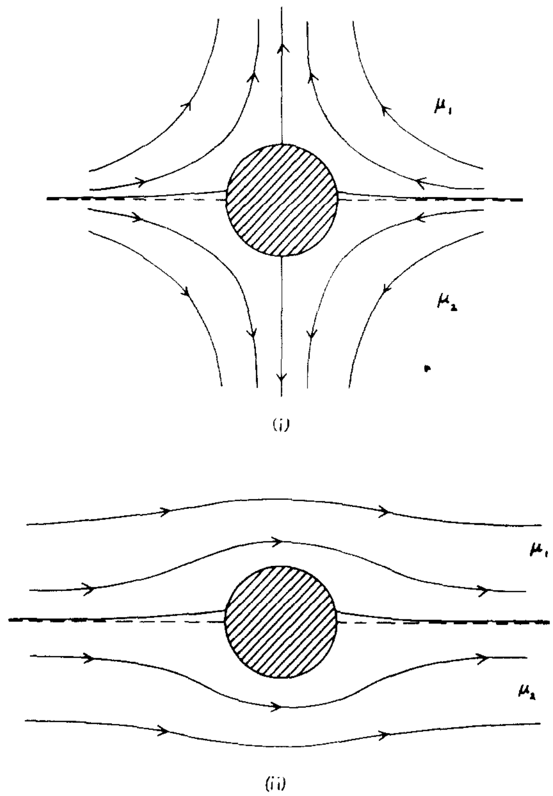

Figure 1. The flow patterns in cases (i) and (ii).

(ii)

$$\frac{q^{(j)}_{1z}}{V} - \frac{3f}{4\rho^2}\left(1-\frac{1}{\rho^2}\right)\cos\phi - \frac{\partial f}{\partial\rho}\left(1-\frac{3}{2\rho}+\frac{1}{2\rho^3}\right)\cos\phi$$

$$+\frac{1}{\rho}\frac{\partial f}{\partial\phi}\left(1-\frac{3}{4\rho}-\frac{1}{4\rho^3}\right)\sin\phi = 0 \quad (j=1,2) \quad \text{at } z=0, \rho>1 \qquad [17]$$

$$[P_1^{(1)} - P_1^{(2)}]_{z=0} = \frac{3V}{\rho^4}\cos\phi + \frac{\gamma}{\mu}\left(\frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}\right) - \frac{\Delta g}{\mu} f \ (\rho > 1). \qquad [18]$$

Since these are the only inhomogeneous conditions on the velocity and pressure fields $\mathbf{q}_1^{(j)}$, $P_1^{(j)}$ $(j = 1, 2)$, it follows that

$$q^{(2)}_{1z}(\rho, \phi, -z) = q^{(1)}_{1z}(\rho, \phi, z): \quad P_1^{(2)}(\rho, \phi, -z) = -P_1^{(1)}(\rho, \phi, z) \qquad [19]$$

for all points such that $z > 0$, $r > 1$. The continuity equation then yields

$$q^{(2)}_{1\rho}(\rho, \phi, -z) = -q^{(1)}_{1\rho}(\rho, \phi, z), \quad q^{(2)}_{1\phi}(\rho, \phi, -z) = -q^{(1)}_{1\phi}(\rho, \phi, z). (z > 0, r > 1). \qquad [20]$$

The tangential stress conditions [14] are identically satisfied but the continuity of velocity condition [11] requires that

$$q^{(j)}_{1\rho} = 0 = q^{(j)}_{1\phi} \quad (j = 1, 2) \quad \text{at } z = 0, \quad \rho > 1. \qquad [21]$$

Thus the problem is reduced to finding $\mathbf{q}_1^{(1)}$, $P_1^{(1)}$ such that the velocity vanishes on the sphere and the tangential velocity components vanish at the interface, where the normal velocity and pressure take related non-zero values.

4. CONSTRUCTION OF THE SOLUTION

An axisymmetric velocity field can be written in terms of a stream function ψ:

$$\mathbf{q} = \text{curl}\left(\frac{\psi}{r \sin\theta}\hat{\phi}\right) = \frac{\hat{r}}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} - \frac{\hat{\theta}}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

where

$$L_{-1}^2\psi = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi = 0 \qquad [22]$$

and the pressure is related to the vorticity by

$$\frac{\partial P}{\partial r} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(L_{-1}\psi), \quad \frac{1}{r}\frac{\partial P}{\partial\theta} = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(L_{-1}\psi). \qquad [23]$$

The no slip condition requires that $\psi = (\partial\psi/\partial r) = 0$ at $r = 1$ and the force exerted by the fluid on the hemisphere $r \leq 1$, $0 \leq \theta \leq \frac{1}{2}\pi$ is

$$2\pi\mu_1\hat{z}\int_0^{\pi/2}\left(-P\cos\theta + \frac{\partial^2\psi}{\partial r^2}\right)_{r=1}\sin\theta\,\mathrm{d}\theta.$$

The pressure integral can be rewritten by applying Green's theorem to the harmonic functions P and $r^{-2}\cos\theta$ in the region $r > 1$, $0 \leq \theta \leq \frac{1}{2}\pi$ and using [23] to write

$$\sin\theta\left(\frac{\partial P}{\partial r}\right)_{r=1} = \frac{\partial}{\partial\theta}\left(\frac{\partial^2\psi}{\partial r^2}\right)_{r=1}.$$

The above force expression becomes

$$\pi\mu_1\hat{z}\left[3\int_0^{\pi/2}\left(\frac{\partial^2\psi}{\partial r^2}\right)_{r=1}\sin\theta\mathrm{d}\theta - \int_1^\infty (P)_{\theta=\frac{1}{2}\pi}\frac{\mathrm{d}r}{r^2}\right].$$

In terms of cylindrical coordinates (ρ, ϕ, z), the above representation of $\mathbf{q}$ in terms of ψ is

$$\mathbf{q} = \text{curl}\left(\frac{\psi}{\rho}\hat{\phi}\right) = \frac{1}{\rho}\left[-\frac{\partial\psi}{\partial z}\hat{\rho} + \frac{\partial\psi}{\partial\rho}\hat{z}\right] \qquad [24]$$

where

$$L_{-1}^2\psi = \left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right)^2\psi = 0$$

$$\frac{\partial P}{\partial\rho} = -\frac{1}{\rho}\frac{\partial}{\partial z}(L_{-1}\psi), \quad \frac{\partial P}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(L_{-1}\psi). \qquad [25]$$

Thus either [19] or [20] implies that any axisymmetric component of $\mathbf{q}_1^{(i)}$ has a stream function $\psi^{(i)}$ which is such that

$$\psi^{(2)}(\rho, -z) = \psi^{(1)}(\rho, z) \quad (z > 0, r > 1) \qquad [26]$$

and, from [21],

$$\frac{\partial \psi^{(j)}}{\partial z} = 0 \quad \text{at } z = 0, \rho > 1 \quad (j = 1.2). \qquad [27]$$

The force exerted on the sphere by this component is given by

$$F = \pi(\mu_1 + \mu_2)\hat{z}\left[3\int_0^{\pi/2}\left(\frac{\partial^2 \psi^{(1)}}{\partial r^2}\right)_{r=1} \sin\theta \, d\theta - \int_1^{\infty} (P^{(1)})_{z=0} \frac{d\rho}{\rho^2}\right]. \qquad [28]$$

In the last integral, the pressure is given in approximation (b) but in case (a) must be eliminated by integrating by parts and using [25] which together with [27] implies that

$$\left(\frac{\partial P^{(1)}}{\partial \rho}\right)_{z=0} = -\frac{1}{\rho}\left(\frac{\partial^3 \psi^{(1)}}{\partial z^3}\right)_{z=0}.$$

Hence an alternative form of [28] is

$$F = 2\pi\mu\hat{z}\left[3\int_0^{\pi/2}\left(\frac{\partial^2 \psi^{(1)}}{\partial r^2}\right)_{r=1} \sin\theta \, d\theta - \int_1^{\infty}\left(\frac{\partial^3 \psi^{(1)}}{\partial z^3}\right)_{z=0} \frac{1}{\rho}\left(1 - \frac{1}{\rho}\right) d\rho\right] \qquad [29]$$

For the fully axisymmetric flow in case (i), $\mathbf{q}_1^{(j)} = \text{curl}\,((1/\rho)\psi^{(j)}\hat{\phi})$ $(j = 1, 2)$ and the interface condition [15] simplifies to

$$\psi^{(j)} = -\tfrac{1}{2}V\left(\rho^2 - \frac{5}{2\rho} + \frac{3}{2\rho^3}\right)f \quad (j = 1, 2) \quad \text{at } z = 0, \rho > 1 \qquad [30]$$

For the non-axisymmetric velocity fields, a representation suitable for spherical boundaries is that used by Brenner & Davis (1981), namely

$$\mathbf{Q}_m = \text{curl}^2[\Psi_m(r, \theta)\cos m\phi\hat{r}] + \text{curl}\,[X_m(r, \theta)\sin m\phi\hat{r}]$$

$$= \left(\frac{\partial^2 \Psi_m}{\partial r^2} - H_m\Psi_m\right)\hat{r}\cos m\phi + \frac{1}{r}\left(\frac{\partial^2 \Psi_m}{\partial r \partial \theta} + \frac{mX_m}{\sin\theta}\right)\hat{\theta}\cos m\phi$$

$$-\frac{1}{r}\left(\frac{m}{\sin\theta}\frac{\partial \Psi_m}{\partial r} + \frac{\partial X_m}{\partial \theta}\right)\hat{\phi}\sin m\phi \qquad [31]$$

$$P_m = \cos m\phi \frac{\partial}{\partial r} H_m\Psi_m \qquad (m \geq 1)$$

where the operator H_m is given by

$$H_m = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) - \frac{m^2}{r^2 \sin^2\theta} \qquad [32]$$

and the functions Ψ_m, X_m satisfy

$$H_m X_m = 0 = H_m^2 \Psi_m \qquad (m \geq 1) \qquad [33]$$

The operator H_1 is related to the Stokes operator L_{-1} by

$$H_1\left(\frac{\psi}{\sin\theta}\right)=\frac{1}{\sin\theta}L_{-1}\psi$$

and hence [22] is equivalent to

$$H_1^2\left(\frac{\psi}{\sin\theta}\right)=0 \tag{34}$$

The no-slip condition on $\mathbf{Q}_m$ requires that

$$\Psi_m=\frac{\partial\Psi_m}{\partial r}=X_m=0 \quad \text{at } r=1 \tag{35}$$

Equations [20] show that components of $\mathbf{q}_i^{(j)}$ of the form $\mathbf{Q}_m$ must be such that

$$\Psi_m^{(2)}(r,\pi-\theta)=-\Psi_m^{(1)}(r,\theta),\quad X_m^{(2)}(r,\pi-\theta)=X_m^{(1)}(r,\theta)(r>1,0\leqslant\theta\leqslant\tfrac{1}{2}\pi) \tag{36}$$

The conditions [21] that the tangential components of $\mathbf{q}_1^{(j)}$ vanish at the interface then imply

$$\left.\begin{aligned} m\frac{\partial\Psi_m^{(j)}}{\partial r}+\frac{\partial X_m^{(j)}}{\partial\theta}&=0\\ \frac{\partial^2\Psi_m^{(j)}}{\partial\theta^2}&=m^2\Psi_m^{(j)}\end{aligned}\right\}\quad (j=1,2)\quad \text{at } \theta=\tfrac{1}{2}\pi, r>1. \tag{37}$$

Since the pressure is given explicitly in terms of Ψ_m in [31], the stress $\boldsymbol{\sigma}_m^{(j)}$ exerted by the fields $\{\mathbf{Q}_m^{(j)}, P_m^{(j)}\}$ on the sphere can, in contrast to the axisymmetric case, be expressed as derivatives of $\Psi_m^{(j)}$ and $X_m^{(j)}$, namely

$$\boldsymbol{\sigma}_m^{(j)}=\mu_j\left[-\frac{\partial}{\partial r}H_m\Psi_m^{(j)}\right]_{r=1}\hat{r}\cos m\phi+\mu_j\left[\frac{\partial^3\Psi_m^{(j)}}{\partial r^2\partial\theta}+\frac{m}{\sin\theta}\frac{\partial X_m^{(j)}}{\partial r}\right]_{r=1}\hat{\theta}\cos m\phi$$
$$-\mu_j\left[\frac{m}{\sin\theta}\frac{\partial^2\Psi_m^{(j)}}{\partial r^2}+\frac{\partial^2X_m^{(j)}}{\partial r\partial\theta}\right]_{r=1}\hat{\phi}\sin m\phi\ (j=1,2,m\geqslant 1).$$

Clearly fields with $m\geqslant 2$ exert zero net force and torque on any cap of the sphere with axis at $\theta=0,\pi$. The force exerted by $\boldsymbol{\sigma}_1^{(1)}$ on the hemisphere $r\leqslant 1, 0\leqslant\theta\leqslant\frac{1}{2}\pi$ is

$$\pi\mu_1\hat{x}\int_0^{\pi/2}\left[-\frac{\partial}{\partial r}H_1\Psi_1^{(1)}\sin\theta+\left(\frac{\partial^3\Psi_1^{(1)}}{\partial r^2\partial\theta}+\frac{1}{\sin\theta}\frac{\partial X_1^{(1)}}{\partial r}\right)\cos\theta\right.$$
$$\left.+\left(\frac{1}{\sin\theta}\frac{\partial^2\Psi_1^{(1)}}{\partial r^2}+\frac{\partial^2X_1^{(1)}}{\partial r\partial\theta}\right)\right]_{r=1}\sin\theta\,\mathrm{d}\theta.$$

On writing down the similar expression for the force exerted by $\boldsymbol{\sigma}_1^{(2)}$ on the hemisphere $r\leqslant 1$, $\frac{1}{2}\pi\leqslant\theta\leqslant\pi$, and using the relations [36], it follows that the force exerted on the sphere by a component $\lambda\mathbf{Q}_1^{(j)}$ of $\lambda\mathbf{q}_1^{(j)}$ is of order λ^2 and therefore negligible in the present approximation. In considering the torque $\mathbf{T}_1$ exerted by $\boldsymbol{\sigma}_1^{(j)}$ on the sphere, the leading contributions from the two hemispheres add and the result is

$$\mathbf{T}_1 = \pi(\mu_1+\mu_2)\hat{y}\int_0^{\pi/2}\left[\left(\frac{1}{\sin\theta}\frac{\partial^2\Psi_1^{(1)}}{\partial r^2}+\frac{\partial^2 X_1^{(1)}}{\partial r\partial\theta}\right)\cos\theta+\frac{\partial^3\Psi_1^{(1)}}{\partial r^2\partial\theta}+\frac{1}{\sin\theta}\frac{\partial X_1^{(1)}}{\partial r}\right]_{r=1}\sin\theta\,d\theta$$

$$= 2\pi\mu\hat{y}\left\{\left[\frac{\partial^2\Psi_1^{(1)}}{\partial r^2}\right]_{\substack{\theta=\frac{1}{2}\pi\\ r=1}}+2\int_0^{\pi/2}\left(\frac{\partial X_1^{(1)}}{\partial r}\right)_{r=1}\sin^2\theta\,d\theta\right\}. \qquad [38]$$

The integral over the hemispherical surface can be transformed into one over the undisturbed interface $z=0$ by applying Green's theorem to the functions $r^{-1}X_1^{(1)}$ and $r^{-2}\sin\theta$ in the region $r\geq 1$, $0\leq\theta\leq\frac{1}{2}\pi$. Then, since

$$\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)\frac{X_1^{(1)}}{r}=\frac{1}{r\sin\theta}L_{-1}[X_1^{(1)}\sin\theta]=\frac{1}{r}H_1(X_1^{(1)})=0,$$

according to (4.12), and

$$\left(\nabla^2-\frac{1}{r^2\sin^2\theta}\right)\frac{\sin\theta}{r^2}=0,$$

it follows, in virtue of [35], that

$$\int_0^{\pi/2}\left(\frac{\partial X_1^{(1)}}{\partial r}\right)_{r=1}\sin^2\theta\,d\theta=\int_1^{\infty}\left(\frac{\partial X_1^{(1)}}{\partial\theta}\right)_{\theta=\frac{1}{2}\pi}\frac{dr}{r^3}.$$

Thus an alternative form of [38], which will be simpler to evaluate is

$$\mathbf{T}_1=2\pi\mu\hat{y}\left\{\left[\frac{\partial^2\Psi_1^{(1)}}{\partial\rho^2}\right]_{\substack{\rho=1\\ z=0}}-2\int_1^{\infty}\left(\frac{\partial X_1^{(1)}}{\partial z}\right)_{z=0}\frac{d\rho}{\rho^2}\right\}. \qquad [39]$$

Since the solutions X_m, Ψ_m of [33] must satisfy conditions at the sphere and at the "interface", their construction requires the use of a coordinate system in which the hemispherical surfaces $r=1$, $z\gtrless 0$ and the plane region $z=0$, $r>1$ belong to the same family of coordinate surfaces. Hence introduce toroidal coordinates by writing

$$\rho=\frac{\sinh\xi}{\cosh\xi-\cos\eta},\quad z=\frac{\sin\eta}{\cosh\xi-\cos\eta} \qquad [40a]$$

or, equivalently

$$r=\left(\frac{\cosh\xi+\cos\eta}{\cosh\xi-\cos\eta}\right)^{1/2},\quad \tan\theta=\frac{\sinh\xi}{\sin\eta}. \qquad [40b]$$

Then, with the interface linearised at $\eta=0$, the fluids of viscosity μ_1, μ_2 occupy respectively the regions $\xi\geq 0$, $0\leq\eta\leq\frac{1}{2}\pi$, $-\pi<\phi\leq\pi$ and $\xi\geq 0$, $-\frac{1}{2}\pi\leq\eta\leq 0$, $-\pi<\phi\leq\pi$.

For each integer $m\geq 0$, it may be shown that

$$H_m[(\cosh\xi+\cos\eta)^{1/2}{}^{\cosh}_{\sinh}s\eta K_s^m(\cosh\xi)]=0 \qquad [41a]$$

$$H_m^2\left[\frac{(\cosh\xi+\cos\eta)^{1/2}}{\cosh\xi-\cos\eta}{}^{\cosh}_{\sinh}s\eta{}^{\cos}_{\sin}\eta K_s^m(\cosh\xi)\right]=0 \qquad [41b]$$

where $K_s^m(\alpha)$ denotes the Mehler conal function of order m, which satisfies, for $\alpha > 1$, the equation

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha^2-1)\frac{\mathrm{d}K_s^m}{\mathrm{d}\alpha}\right]+\left[s^2+\tfrac{1}{4}-\frac{m^2}{\alpha^2-1}\right]K_s^m=0 \qquad [42]$$

and is defined as an associated Legendre function by

$$K_s^m = P^m_{-\frac{1}{2}+is} \quad (s \geqslant 0, m \geqslant 0).$$

Although [41b] is valid for $m = 0$ and [24] takes the form [31] when $\psi/\sin\theta$ is replaced by $-\partial\Psi_0/\partial\theta$, the velocity representation is inappropriate for axisymmetric flows because the function $K_s^m(\cosh\xi)$ in [41b], which for $m \geqslant 1$ vanishes on the axis $\xi = 0$, does not do so for $m = 0$. For small α,

$$K_s(\alpha) \sim 1-\tfrac{1}{2}(s^2+\tfrac{1}{4})(\alpha-1)+\ldots \qquad [43]$$

The solution for $\psi^{(1)}$, which satisfies [34] in the axisymmetric region $\xi > 0, 0 \leqslant \eta \leqslant \frac{1}{2}\pi$, and the conditions $\psi^{(1)} = \partial\psi^{(1)}/\partial n = 0$ at $\eta = \frac{1}{2}\pi$, is of the form

$$\psi^{(1)} = \frac{V\sqrt{(2)}\sinh\xi}{(\cosh\xi-\cos\eta)^{3/2}}\int_0^\infty \left\{A(s)\cos\eta\,\frac{\sinh s(\frac{1}{2}\pi-\eta)}{\sinh\frac{1}{2}s\pi}\right.$$

$$\left.-B(s)\left[\sin\eta\,\frac{\sinh s\,(\frac{1}{2}\pi-\eta)}{\cosh\frac{1}{2}s\pi}-2s\cos\eta\,\frac{\sinh s\eta}{\sinh s\pi}\right]\right\}K_s^{1}(\cosh\xi)\,\mathrm{d}s \qquad [44]$$

by substitution of [40b] for $\sin\theta$ and suitable superposition of solutions given by [41b]. Since, from [A1],

$$K_s^1(\cosh\xi) = \sinh\xi\; K_s'(\cosh\xi)$$

and ρ is given by [40a], it is readily seen that [44] is a solution of the type constructed by Payne & Pell (1960). A relation between $A(s)$ and $B(s)$ is obtained by applying condition [27] which requires that

$$\frac{\partial\psi^{(1)}}{\partial\eta} = 0 \quad \text{at } \eta = 0 \quad (\xi > 0)$$

Thus

$$sA(s)\cosh^2\tfrac{1}{2}s\pi = B(s)[s^2-\sinh^2\tfrac{1}{2}s\pi] \qquad [45]$$

In terms of the toroidal coordinates ξ,η defined by [40a,b] with $\alpha = \cosh\xi$, the force expression can be written in the form

$$\mathbf{F} = 2\pi\mu\hat{z}\left\{3\int_1^\infty\left(\frac{\partial^2\psi^{(1)}}{\partial\eta^2}\right)_{\eta=\frac{1}{2}\pi}\mathrm{d}\alpha-\int_1^\infty(\alpha-1)^3\left(\frac{\partial^3\psi^{(1)}}{\partial\eta^3}\right)_{\eta=0}\left[1-\left(\frac{\alpha-1}{\alpha+1}\right)^{1/2}\right]\frac{\mathrm{d}\alpha}{\alpha^2-1}\right\}.$$

since $(\psi^{(1)})_{\eta=\frac{1}{2}\pi} \equiv 0 \equiv (\partial\psi^{(1)}/\partial\eta)_{\eta=0}$. Substitution of [44] then yields

$$\mathbf{F} = 2\pi\mu V\hat{z}\left\{6\surd(2)\int_1^\infty \frac{\alpha^2-1}{\alpha^{3/2}}\int_0^\infty \frac{s[A(s)-sB(s)]}{\sinh\frac{1}{2}s\pi}K_s'(\alpha)\,\mathrm{d}s\,\mathrm{d}\alpha\right.$$

$$\left.+\surd(2)\int_1^\infty (\alpha-1)^{3/2}\left[1-\left(\frac{\alpha-1}{\alpha+1}\right)^{1/2}\right]\int_0^\infty \frac{(s^2+1)sA(s)\sinh s\pi}{s^2-\sinh^2\frac{1}{2}s\pi}K_s'(\alpha)\,\mathrm{d}s\,\mathrm{d}\alpha\right\}$$

$$= -8\pi\mu V\hat{z}\left\{3\int_0^\infty [A(s)-sB(s)]\frac{s^2+\frac{1}{4}}{\sinh^2\frac{1}{2}s\pi}\,\mathrm{d}s\right.$$

$$\left.+\int_0^\infty \frac{(s^2+1)A(s)}{s^2-\sinh^2\frac{1}{2}s\pi}\left[2(s^2+\tfrac{1}{4})+\frac{(s^2+\frac{1}{4})(\cosh s\pi+1)}{2(s^2+1)}-\tfrac{1}{2}\cosh s\pi\right]\mathrm{d}s\right\}$$

by use of [A6b] and [A8b]. Then, with $B(s)$ eliminated by [45], the expression for $\mathbf{F}$ can be reduced to

$$\mathbf{F} = 2\pi\mu V\hat{z}\left[2\int_0^\infty \frac{A(s)(s^2+1)(2s^2+1)}{s^2-\sinh^2\frac{1}{2}s\pi}\,\mathrm{d}s - 3\int_0^\infty A(s)\,\mathrm{d}s\right]. \qquad [46]$$

Solutions for $\Psi_m^{(1)}$ and $X_m^{(1)}$ $(0<\eta<\frac{1}{2}\pi)$ satisfying [33] and [34] are given, from [41a, b], by

$$\Psi_m^{(1)} = V\surd(2)\frac{(\cosh\xi+\cos\eta)^{1/2}}{\cosh\xi-\cos\eta}\int_0^\infty\left\{C(s)\cos\eta\frac{\sinh s(\frac{1}{2}\pi-\eta)}{\sinh\frac{1}{2}s\pi}\right.$$

$$\left.-D(s)\left[\sin\eta\frac{\sinh s\,(\frac{1}{2}\pi-\eta)}{\cosh\frac{1}{2}s\pi}-2s\cos\eta\frac{\sinh s\eta}{\sinh s\pi}\right]\right\}K_s^m(\cosh\xi)\,\mathrm{d}s \qquad [47]$$

$$X_m^{(1)} = V\surd(2)(\cosh\xi+\cos\eta)^{1/2}\int_0^\infty E(s)\frac{\sinh s\,(\frac{1}{2}\pi-\eta)}{\cosh\frac{1}{2}s\pi}K_s^m(\cosh\xi)\,\mathrm{d}s. \qquad [48]$$

Here the functions $C(s)$, $D(s)$ and $E(s)$ are connected by the two conditions [37], which in terms of ξ, η take the form

$$\left.\begin{aligned} &m\left(\frac{\cosh\xi-1}{\sinh\xi}\right)\frac{\partial\Psi_m^{(j)}}{\partial\xi}+\frac{\partial X_m^{(j)}}{\partial\eta}=0\\ &\sinh^2\xi\frac{\partial^2\Psi_m^{(j)}}{\partial\eta^2}-\sinh\xi\cosh\xi\frac{\partial\Psi_m^{(j)}}{\partial\xi}=m^2\Psi_m^{(j)}\end{aligned}\right\}(j=1,2)\text{ at }\eta=0,\ \xi>0. \qquad [49]$$

5. STRONG SURFACE TENSION OR GRAVITY FORCES

In approximations (a) and (b), in which either γ or Δg is large compared with μV, the dominant contribution to the velocity field $\mathbf{q}_1^{(j)}$ can be calculated by ignoring in [13] the discontinuity in the pressure field $P_1^{(j)}$. Equivalently, the surface tension and gravity forces exactly balance, in these approximations, the normal stress discontinuity due to the introduction of the different viscosities into the zero order flow $\mathbf{q}_0$. Thus, in the axisymmetric case (i), [16] yields

$$F=\frac{9\mu V}{\gamma}\left\{I_0\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\int K_0\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\frac{\mathrm{d}\rho}{\rho^4}-K_0\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\int I_0\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\frac{\mathrm{d}\rho}{\rho^4}\right\}(\rho>1) \qquad [50]$$

whilst in the asymmetric case (ii), [18] implies that

$$f=-\frac{3\mu V}{\gamma}\left\{I_1\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\int K_1\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\frac{\mathrm{d}\rho}{\rho^3}-K_1\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\int I_1\left(\rho\sqrt{\left(\frac{\Delta g}{\gamma}\right)}\right)\frac{\mathrm{d}\rho}{\rho^3}\right\}\cos\phi$$

$$(\rho>1). \qquad [51]$$

Here the constants in the indefinite integrals are all set equal to zero because I_0, I_1 increase exponentially with ρ, whilst K_0, K_1 decay too slowly with increasing ρ in approximation (a) and are exponentially small in approximation (b).

It is the need to simplify expressions [50] and [51] which restricts detailed consideration here to sufficiently large or small values of $\Delta g/\gamma$. In approximation (a), the simplifications, obtained by using the leading term of each Bessel function at small values of the argument, i.e. $\rho^2 \ll 4\gamma/\Delta g$, are

$$f = \frac{\mu V}{\gamma\rho^3} \quad (\rho > 1) \tag{50a}$$

$$f = -\frac{\mu V}{\gamma\rho^2}\cos\phi \quad (\rho > 1). \tag{51a}$$

The above outer limit on ρ may evidently be regarded as immaterial and therefore ignored in the subsequent analysis. Similarly, in approximation (b), the simplifications of (5.1), (5.2) obtained at large values of the argument $\rho(\Delta g/\gamma)^{1/2}$ are

$$f = -\frac{9\mu V}{\Delta g\rho^5} \quad (\rho > 1) \tag{50b}$$

$$f = \frac{3\mu V}{\Delta g\rho^4}\cos\phi \quad (\rho > 1) \tag{51b}$$

where the extension of validity down to the value unity of ρ requires $\Delta g \gg 10\gamma$. Note that the interface is flatter when surface tension dominates than when gravity does so.

On considering first case (i), substitution of [50a] into [15] shows that

$$q_{1z}^{(1)} = -\frac{V}{2\rho}\frac{d}{d\rho}\left\{\left(\rho^2 - \frac{5}{2\rho} + \frac{3}{2\rho^3}\right)\frac{\mu V}{\gamma\rho^3}\right\} \quad (z = 0, \rho > 1)$$

and hence, from [24], the stream function is such that

$$\psi^{(1)}(\rho, 0) = -\frac{\mu V^2}{2\gamma}\left(\frac{1}{\rho} - \frac{5}{2\rho^4} + \frac{3}{2\rho^6}\right) \quad (\rho > 1) \tag{52}$$

With $\psi^{(1)}$ given by [44], condition [52] yields an equation determining $A(s)$, namely:

$$\surd(2)(\alpha - 1)^{1/2}\rho^2\int_0^\infty A(s)K_s'(\alpha)\,ds = -\frac{\mu V}{2\gamma}\left(\frac{1}{\rho} - \frac{5}{2\rho^4} + \frac{3}{2\rho^6}\right)$$

where $\alpha = \cosh\xi$. A rearrangement of this equation, which enables the integrals given in appendices 1, 2 to be readily used, is

$$\surd(2)\frac{\gamma}{\mu V}\int_0^\infty A(s)K_s'(\alpha)\,ds = \frac{1}{2(\alpha-1)^{1/2}}\left\{\left(\frac{1}{\rho^2} - \frac{1}{\rho^3}\right) - \frac{1}{2}\left(1 - \frac{1}{\rho^2}\right) + \frac{1}{2}\left(1 - \frac{1}{\rho^2}\right)^2\left(1 - \frac{1}{\rho^2} - \frac{3}{\rho^4}\right)\right\}$$
$$= \frac{1}{2}\left[\frac{(\alpha-1)^{1/2}}{\alpha+1} - \frac{\alpha-1}{(\alpha+1)^{3/2}}\right] - \frac{1}{2(\alpha+1)(\alpha-1)^{1/2}} + \frac{1}{(\alpha+1)^2(\alpha-1)^{1/2}}\left[1 - \left(\frac{\alpha-1}{\alpha+1}\right) - 3\left(\frac{\alpha-1}{\alpha+1}\right)^2\right]$$

The inversion is achieved by means of [A8a], [A9a] and [A12], which show that

$$\frac{\gamma}{\mu V}A(s) = a(s) + \frac{1}{s^2+\frac{1}{4}}[\tfrac{1}{4} + J_{0,1}(s) - J_{1,1}(s) - 3J_{2,1}(s)]$$

$$= -\frac{2}{\cosh s\pi} - \frac{\cosh s\pi + 1}{2(s^2+1)\cosh s\pi} + \frac{1}{s^2+\frac{1}{4}}\left[\frac{3}{4} - \frac{1}{2}\int_0^\infty \frac{s\sin 2vs}{\cosh^3 v}(1 - \tfrac{3}{2}\tanh^2 v - \tfrac{45}{8}\tanh^4 v)\,dv\right].$$

after substituting [A15].

With $A(s)$ determined, the corresponding force $\mathbf{F}$ is given by [46]. The integrals [A16] and [A17] show that

$$\frac{\gamma}{\mu V}\int_0^\infty A(s)\,\mathrm{d}s = -2+\tfrac{3}{4}\pi-\frac{\pi}{4}\int_0^\infty \frac{e^{-v}}{\cosh^3 v}(1-\tfrac{3}{2}\tanh^2 v-\tfrac{45}{8}\tanh^4 v)\,\mathrm{d}v$$

$$= -2+\tfrac{45}{64}\pi$$

by integration with respect to tanh v. When the force $\lambda\mathbf{F}$ due to the field $\lambda\mathbf{q}_1^{(j)}$ is added to the force $3\pi(\mu_1-\mu_2)V\hat{z}$ due to the velocity field $\mathbf{q}_0$, it is seen that, to the order calculated:

$$\text{Force on sphere} = 3\pi(\mu_1-\mu_2)V\hat{z}\left\{1-\frac{\mu V}{\gamma}[\tfrac{45}{64}\pi-2]-\tfrac{2}{3}\int_0^\infty\frac{A(s)(s^2+1)(2s^2+1)}{\sinh^2\frac{1}{2}s\pi-s^2}\mathrm{d}s\right\}. \quad [54]$$

Similarly, substitution of [50b] into [15] yields

$$\sqrt{(2)}(\alpha-1)^{1/2}\rho^2\int_0^\infty A(s)K_s'(\alpha)\,\mathrm{d}s = \frac{q\mu V}{2\Delta g}\left(\frac{1}{\rho^3}-\frac{5}{2\rho^6}+\frac{3}{2\rho^8}\right)$$

and hence

$$\sqrt{(2)}\frac{\Delta g}{9\mu V}\int_0^\infty A(s)K_s'(\alpha)\,\mathrm{d}s = \frac{-1}{2(\alpha-1)^{1/2}}\left\{\left(\frac{1}{\rho^2}-\frac{1}{\rho^5}\right)-\tfrac{3}{2}\left(1-\frac{1}{\rho^2}\right)+\tfrac{1}{2}\left(1-\frac{1}{\rho^2}\right)^2\left(3+\frac{1}{\rho^2}-\frac{1}{\rho^4}-\frac{3}{\rho^6}\right)\right\}$$

$$= -\tfrac{1}{2}\left[\frac{(\alpha-1)^{1/2}}{\alpha+1}-\frac{\alpha-1}{(\alpha+1)^{3/2}}\right]+\frac{\alpha-1}{(\alpha+1)^{5/2}}+\frac{3}{2(\alpha+1)(\alpha-1)^{1/2}}$$

$$-\frac{1}{(\alpha+1)^2(\alpha-1)^{1/2}}\left[3+\left(\frac{\alpha-1}{\alpha+1}\right)-\left(\frac{\alpha-1}{\alpha+1}\right)^2-3\left(\frac{\alpha-1}{\alpha+1}\right)^3\right].$$

The inversion, which now requires the additional use of [A6a] and the α-derivative of (A5a) at $\eta=\pi$, then yields

$$\frac{\Delta g}{9\mu V}A(s) = \frac{4(1-\frac{2}{3}s^2)}{\cosh s\pi}+\frac{\cosh s\pi+1}{2(s^2+1)\cosh s\pi}$$

$$-\frac{1}{s^2+\frac{1}{4}}\left[\tfrac{5}{4}-\tfrac{3}{2}\int_0^\infty\frac{s\sin 2vs}{\cosh^3 v}(1+\tfrac{1}{2}\tanh^2 v-\tfrac{5}{8}\tanh^4 v-\tfrac{35}{16}\tanh^6 v)\,\mathrm{d}v\right],$$

corresponding to [53]. Thus

$$\int_0^\infty A(s)\,\mathrm{d}s = \frac{\mu V}{\Delta g}(24-\tfrac{4095}{512}\pi)$$

and the force expression [54] is accordingly modified for approximation (b).

In case (ii), substitution of [51a] into [17] shows, after some rearrangement, that

$$\frac{q_{1z}^{(1)}}{V} = -\frac{\mu V}{2\gamma\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho}-\frac{3}{2\rho^2}+\frac{1}{2\rho^4}\right)+\frac{3\mu V}{2\gamma\rho^3}\left(1-\frac{3}{\rho}+\frac{1}{2\rho^3}\right)\cos 2\phi. \quad [55]$$

Evidently the velocity field $\mathbf{q}_1^{(j)}$ exerts, in this approximation, no torque on the sphere but only a force due to its axisymmetric component which, from [24], may be derived from a stream function $\hat{\psi}^{(1)}(\rho, z)$ such that

$$\hat{\psi}^{(1)}(\rho, 0) = -\frac{\mu V^2}{2\gamma}\left(\frac{1}{\rho} - \frac{3}{2\rho^2} + \frac{1}{2\rho^4}\right) \quad (\rho > 1). \tag{56}$$

The calculation of $\hat{\psi}^{(1)}$ is similar to that of $\psi^{(1)}$ and the details are remarkably alike. Since, from [52] and [56],

$$\hat{\psi}^{(1)}(\rho, 0) - \psi^{(1)}(\rho, 0) = \frac{3\mu V^2}{4\gamma\rho^2}\left(1 - \frac{1}{\rho^2}\right)^2 \quad (\rho > 1),$$

it follows, in an obvious notation, that

$$\sqrt{(2)}\frac{\gamma}{\mu V}\int_0^\infty [\hat{A}(s) - A(s)]K_s'(\alpha)\,\mathrm{d}s = 3\frac{(\alpha-1)^{3/2}}{(\alpha+1)^4}.$$

The inversion is given by [A12] and subsequent substitution of [A15] shows that

$$\frac{\gamma}{\mu V}[\hat{A}(s) - A(s)] = \frac{3}{s^2+\frac{1}{4}}J_{2,1}(s) = -\frac{45}{16(s^2+\frac{1}{4})}\int_0^\infty \frac{s\sin 2vs}{\cosh^3 v}\tanh^4 v\,\mathrm{d}v. \tag{57}$$

Thus $\hat{A}(s)$ can be derived from $A(s)$ by simply deleting the $J_{2,1}$ term from [53]. Hence, by comparison with case (i),

$$\frac{\gamma}{\mu V}\int_0^\infty \hat{A}(s)\,\mathrm{d}s = -2 + \tfrac{21}{32}\pi$$

and, to the order calculated:

Force on sphere

$$= 3\pi(\mu_1+\mu_2)V\hat{x} - 3\pi(\mu_1-\mu_2)V\hat{z}\left[\frac{\mu V}{\gamma}(\tfrac{21}{32}\pi - 2) + \tfrac{2}{3}\int_0^\infty \frac{\hat{A}(s)(s^2+1)(2s^2+1)}{\sinh^2\frac{1}{2}s\pi - s^2}\mathrm{d}s\right] \tag{58}$$

$$\text{Torque on sphere} = \tfrac{3}{2}\pi(\mu_1-\mu_2)V\hat{y} \tag{59}$$

Similarly, for approximation (b), substitution of [51b] into [17] shows that

$$\frac{q_{1z}^{(1)}}{V} = \frac{3\mu V}{2\Delta g\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^3} - \frac{3}{2\rho^4} + \frac{1}{2\rho^6}\right) - \frac{15\mu V}{2\Delta g\rho^5}\left(1 - \frac{3}{2\rho} + \frac{1}{2\rho^3}\right)\cos 2\phi$$

which expression differs little from its counterpart [55] in approximation (a). Then

$$\tfrac{1}{3}\hat{\psi}^{(1)}(\rho, 0) - \tfrac{1}{9}\psi^{(1)}(\rho, 0) = -\frac{3\mu V^2}{4\Delta g\rho^4}\left(1 - \frac{1}{\rho^2}\right)^2$$

and hence

$$\frac{\Delta g}{\mu V}[\tfrac{1}{3}\hat{A}(s) - \tfrac{1}{9}A(s)] = -\frac{3}{s^2+\frac{1}{4}}J_{3,1}(s) = \frac{105}{32(s^2+\frac{1}{4})}\int_0^\infty \frac{\sin 2vs}{\cosh^3 v}\tanh^6 v\,\mathrm{d}v$$

Thus $\frac{1}{3}\hat{A}(s)$ can be derived by deleting the $J_{3,1}$ term in $\frac{1}{9}A(s)$ and so, by further comparison with (i),

$$\int_0^\infty \hat{A}(s)\,\mathrm{d}s = \frac{\mu V}{\Delta g}\left(8 - \tfrac{165}{64}\pi\right)$$

with a consequent adjustment to [58] for approximation (b).

6. WEAK SURFACE TENSION AND GRAVITY FORCES

With γ and Δg both small compared with μV, approximation (c) involves the discarding of the surface tension terms in [13], with the result that the field $\mathbf{q}_1^{(j)}$ is determined by a prescribed pressure discontinuity at the "interface". Thus in the axisymmetric case (i), [16] yields

$$[P_1^{(1)}]_{z=0} = -\frac{9V}{2\rho^5} \quad (\rho > 1) \tag{60}$$

whilst in the asymmetric case (ii), [18] implies that

$$[P_1^{(1)}]_{z=0} = \frac{3V}{2\rho^4}\cos\phi \quad (\rho > 1) \tag{61}$$

In both [60] and [61], $[P_1^{(2)}]_{z=0}$ has been eliminated by use of [19], which states that the pressure field $P_1^{(j)}$ is an odd function of z.

Again considering case (i) first, condition [60] can be replaced by one prescribing a derivative of the stream function $\psi^{(1)}$ because [25] and [27] imply that

$$\frac{\partial}{\partial\rho}[P_1^{(1)}]_{z=0} = -\frac{1}{\rho}\left(\frac{\partial^3\psi^{(1)}}{\partial z^3}\right)_{z=0}$$

and hence

$$\left(\frac{\partial^3\psi^{(1)}}{\partial z^3}\right)_{z=0} = -\frac{45V}{2\rho^5} \quad (\rho > 1). \tag{62}$$

Then substitution of [44] in [62] yields an equation determining $A(s)$, namely

$$-\frac{45}{2}\left(\frac{\alpha-1}{\alpha+1}\right)^{5/2} = \frac{1}{V}(\alpha-1)^3\left(\frac{\partial^3\psi^{(1)}}{\partial\eta^3}\right)_{\eta=0}$$
$$= -\sqrt{(2)}(\alpha-1)^{3/2}(\alpha^2-1)\int_0^\infty \frac{(s^2+1)sA(s)\sinh s\pi}{s^2-\sinh^2\frac{1}{2}s\pi}K_s'(\alpha)\,\mathrm{d}s.$$

This can be integrated with respect to α, showing that

$$\frac{9}{(\alpha+1)^{5/2}} = -\sqrt{(2)}\int_0^\infty \frac{(s^2+1)sA(s)\sinh s\pi}{s^2-\sinh^2\frac{1}{2}s\pi}K_s(\alpha)\,\mathrm{d}s.$$

The inversion for $A(s)$ is achieved by combining (A4a) suitably with the η-derivative of (A5a) and letting $\eta \to \pi$. Thus

$$A(s) = \frac{4s(\sinh^2\frac{1}{2}s\pi - s^2)}{\sinh s\pi\cosh s\pi} \tag{63}$$

Substitution of [63] in [46] shows that the force on the sphere due to the velocity field $\lambda \mathbf{q}_1^{(i)}$ is, in this approximation

$$\lambda \mathbf{F} = -\pi(\mu_1-\mu_2)V\hat{z}\int_0^\infty \left[\frac{8(s^2+\frac{1}{4})(2s^2+1)}{\cosh s\pi}-6\right]\frac{s\,\mathrm{d}s}{\sinh s\pi}$$

$$= -\tfrac{33}{16}\pi(\mu_1-\mu_2)V\hat{z}$$

by use of [A20]. Hence the force $3\pi(\mu_1-\mu_2)V\hat{z}$ due to the zero order field $\mathbf{q}_0$ is reduced by $\lambda\mathbf{F}$ to $\frac{15}{16}\pi(\mu_1-\mu_2)V\hat{z}$.

Although the force evaluation is simpler here than for small $\mu V/\gamma$, the calculation of the interface displacement function $f(\rho)$ is considerably more lengthy. As in section 5, [15], [24] and [44] imply

$$-\tfrac{1}{2}\left(\rho^2-\frac{5}{2\rho}+\frac{3}{2\rho^3}\right)f(\rho)=\frac{\sqrt{(2)}(\alpha^2-1)}{(\alpha-1)^{3/2}}\int_0^\infty A(s)K_s'(\alpha)\,\mathrm{d}s$$

i.e.

$$\left(1-\frac{5}{2\rho^3}+\frac{3}{2\rho^5}\right)f(\rho)=-2\sqrt{(2)}\,(\alpha-1)^{1/2}\int_0^\infty A(s)K_s'(\alpha)\,\mathrm{d}s. \qquad [64]$$

Now [63] and the integrals [A22] and [A23] show that

$$-\sqrt{(2)}\int_0^\infty A(s)K_s'(\alpha)\,\mathrm{d}s=-2\sqrt{(2)}\int_0^\infty\frac{s(\cosh s\pi-1-2s^2)}{\sinh s\pi\cosh s\pi}K_s'(\alpha)\,\mathrm{d}s$$

$$=\frac{6}{\pi}\int_0^\infty\frac{\mathrm{d}u}{(\cosh u+1)(\cosh u+\alpha)^{5/2}}.$$

Then the substitution

$$t=\frac{2}{\cosh u+1},\quad \frac{\mathrm{d}t}{\mathrm{d}u}=-t\sqrt{(1-t)} \qquad [65]$$

yields the alternative expression

$$-\sqrt{(2)}\int_0^\infty A(s)K_s'(\alpha)\,\mathrm{d}s=\frac{3}{\pi}\left(\frac{\rho^2-1}{2}\right)^{5/2}\int_0^1\frac{t^{5/2}\,\mathrm{d}t}{(1-t)^{1/2}(\rho^2-1+t)^{5/2}}$$

so [64] becomes, after cancelling a factor $(\rho-1)^2$,

$$\left(1+\frac{2}{\rho}+\frac{3}{\rho^2}+\frac{3}{2\rho^3}\right)f=\frac{3\rho^2}{2\pi}(\rho+1)^2\int_0^1\left(\frac{t}{\rho^2-1+t}\right)^{5/2}\frac{\mathrm{d}t}{(1-t)^{1/2}}. \qquad [66]$$

Evidently, as $\rho\to\infty$,

$$f(\rho)\sim\frac{3}{2\pi\rho}\int_0^1\frac{t^{5/2}\,\mathrm{d}t}{(1-t)^{1/2}}+0\left(\frac{1}{\rho^3}\right)$$

$$=\frac{15}{32\rho}+0\left(\frac{1}{\rho^3}\right) \qquad [67]$$

and comparison with [50] shows that the displacement of the interface tends to zero at large distance much more slowly than in either of approximations (a) and (b).

The behaviour of $f(\rho)$ in the $\rho \to 1$ limit is found by using the following integral representation of the hypergeometric function:

$$F(a, b, c, z) = [B(b, c-b)]^{-1} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

where $B(x, y)$ denotes the Beta function, $c > b > 0$ and z is not real and positive. Asymptotic forms of F for large z, valid for non-integral values of $(c-b)$, are given by Abramovitz & Stegun (1964, section 15.3.13/14) for a and b being equal or differing by an integer. In [66], $b - a = 1$ and this particular case emerges, after considerable manipulation, to be also the only one required when the asymmetric case (ii) is subsequently considered. Thus the following simplified asymptotic form is sufficient here:

$$\int_0^1 \left(\frac{t}{\rho^2 - 1 + t}\right)^{k-1/2} (1-t)^{l-(1/2)}\,dt$$

$$= (\rho^2 - 1)^{-(k-(1/2))} B(k + \tfrac{1}{2}, l + \tfrac{1}{2}) F\left(k - \tfrac{1}{2}, k + \tfrac{1}{2}, k + l + 1, -\frac{1}{\rho^2 - 1}\right)$$

$$\sim \frac{1}{l + \frac{1}{2}} - (k - \tfrac{1}{2})(\rho^2 - 1)\log_e\left(\frac{1}{\rho^2 - 1}\right) + 0(\rho^2 - 1) \qquad [68]$$

where k and l are non-negative integers. Hence the form of $f(\rho)$, given by [66], in the $\rho \to 1$ limit is

$$f(\rho) \sim \frac{4}{5\pi}\left[2 - \tfrac{5}{2}(\rho^2 - 1)\log_e\left(\frac{1}{\rho^2 - 1}\right) + 0(\rho^2 - 1)\right] \qquad [69]$$

$$\sim \frac{8}{5\pi} - \frac{4}{\pi}(\rho - 1)\log_e\left(\frac{1}{\rho - 1}\right) + 0(\rho - 1).$$

So although $f(1)$ is finite, $f'(\rho)$ becomes logarithmically infinite as $\rho \to 1$, corresponding to tangential contact of the interface with the sphere and complete "wetting" by the less viscous fluid. This angle of contact satisfies trivially the equations given by Michael (1964) for the Stokes flow of adjoining viscous fluids at a corner. The linearisation of the interface conditions at $z = 0$ depends only on f remaining bounded but those conditions involving the direction of the normal $\hat{n}$, given by [9], assume that the derivatives of f are bounded. Condition [12] is a differential equation for $f(\rho)$ with a regular singular point at $\rho = 1$, whilst, since the relevant velocity derivatives tend to zero as $\rho \to 1$, condition [13] (with $\gamma = 0$) remains appropriate for the pressure discontinuity in $P_1^{(j)}$. It is in this "weak" sense that the solution obtained is valid.

Proceeding now to consider case (ii), condition [61] shows that in the absence of surface tension forces, a suitable representation of the velocity and pressure fields $\mathbf{q}_1^{(j)}$ and $P_1^{(j)}$ is given by [31], with m set equal to unity. Evidently [61] implies

$$[H_1\Psi_1^{(1)}]_{z=0} = -\frac{V}{2\rho^3}.$$

But the second of conditions [37] or, more directly, the vanishing $\hat{r}$ component of [31] implies that

$$[H_1\Psi_1^{(1)}]_{z=0} = \left[\frac{\partial^2\Psi_1^{(1)}}{\partial\rho^2}\right]_{z=0}$$

and hence, after two integrations and use of conditions [35], the inhomogeneous condition on $\Psi_1^{(1)}$ is found to be

$$[\Psi_1^{(1)}]_{z=0} = -\frac{V}{4\rho}(\rho-1)^2. \qquad [70]$$

Then the first of conditions [37] shows that $X_1^{(1)}$ is related to $\Psi_1^{(1)}$ by the condition

$$\left(\frac{\partial X_1^{(1)}}{\partial z}\right)_{z=0} = \frac{1}{\rho}\frac{\partial}{\partial\rho}[\Psi_1^{(1)}]_{z=0} = -\frac{V}{4\rho}\left(1-\frac{1}{\rho^2}\right). \qquad [71]$$

The torque on the sphere due to $\mathbf{q}_1^{(j)}$ is then easily evaluated by direct substitution of [70] and [71] into [39]. Thus

$$\mathbf{T}_1 = 2\pi\mu V\hat{y}\left[-\tfrac{1}{2}+\tfrac{1}{2}\int_1^\infty\left(1-\frac{1}{\rho^2}\right)\frac{d\rho}{\rho^3}\right] = -\tfrac{3}{4}\pi\mu V\hat{y}.$$

When $\lambda\mathbf{T}_1$ is added to the torque $\frac{3}{2}\pi(\mu_1-\mu_2)V\hat{y}$ exerted by the zero order flow $\mathbf{q}_0$, the total torque up to the order calculated is found to be $\frac{9}{8}(\mu_1-\mu_2)V\hat{y}$.

Solutions for $\Psi_1^{(1)}$ and $X_1^{(1)}$ are given by [47] and [48] and the functions $C(s)$, $D(s)$ and $E(s)$ are determined by the conditions [70], [71] and the second of [49]. From [47] and [70],

$$\sqrt{(2)}\frac{(\alpha+1)^{1/2}}{\alpha-1}\int_0^\infty C(s)K_s^{1}(\alpha)\,ds = \tfrac{1}{2}\left[1-\frac{\alpha}{(\alpha^2-1)^{1/2}}\right] \qquad [72a]$$

whence, using [A1],

$$\int_0^\infty C(s)K_s'(\alpha)\,ds = \frac{1}{2\sqrt{(2)}}\left[\frac{(\alpha-1)^{1/2}}{\alpha+1}-\frac{\alpha}{(\alpha+1)^{3/2}}\right].$$

Then, by comparison with the inversion for $a(s)$ in [A8 a, b], it follows that

$$C(s) = -\frac{1}{\cosh s\pi}-\frac{\cosh s\pi+1}{2(s^2+1)\cosh s\pi}+\frac{1}{2(s^2+\frac{1}{4})} \qquad [72b]$$

Also, from [48] and [71],

$$\frac{(\alpha-1)^{1/2}}{2(\alpha+1)^{3/2}} = -(\alpha-1)\left(\frac{\partial X_1^{(1)}}{\partial\eta}\right)_{\eta=0} = \sqrt{(2)}(\alpha-1)(\alpha+1)^{1/2}\int_0^\infty sE(s)K_s^{1}(\alpha)\,ds$$

whence

$$\frac{1}{2\sqrt{(2)}}(\alpha+1)^{-3/2} = (\alpha^2-1)\int_0^\infty sE(s)K_s'(\alpha)\,ds. \qquad [73a]$$

The identity [A10a] now implies that

$$\int_0^\infty [sE(s) - \tfrac{1}{8}b(s)]K_s'(\alpha)\,\mathrm{d}s = -\frac{1}{4\sqrt{(2)}}(\alpha+1)^{-5/2}$$

which, after an integration with respect to α, yields

$$\int_0^\infty [sE(s) - \tfrac{1}{8}b(s)]K_s(\alpha)\,\mathrm{d}s = \frac{1}{6\sqrt{(2)}}(\alpha+1)^{-3/2}.$$

Then [A5a] provides a simple inversion which, with the substitution of [A10b] for $b(s)$, shows that

$$E(s) = -\frac{\tanh s\pi}{8(s^2+\frac{1}{4})} + \frac{s}{\cosh s\pi}\left[\frac{1}{3} + \frac{1}{4(s^2+\frac{1}{4})}\right]. \qquad [73b]$$

Lastly the determination of $D(s)$ is achieved by first substituting [48] in the second of conditions [49]. Thus

$$(\alpha+1)^{3/2}\int_0^\infty [(s^2-1)C(s) + 2sD(s)]\,K_s^{\,1}(\alpha)\,\mathrm{d}s - \left[\tfrac{1}{2}(\alpha+1)^{1/2} + \frac{(\alpha+1)^{3/2}}{\alpha-1}\right]\int_0^\infty C(s)K_s^{\,1}(\alpha)\,\mathrm{d}s$$
$$= \left[(\alpha^2-1)\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha} + 1\right]\frac{(\alpha+1)^{1/2}}{\alpha-1}\int_0^\infty C(s)K_s^{\,1}(\alpha)\,\mathrm{d}s$$

where all but the first integral can be expressed in closed form by means of [72a]. The resulting simplified form is

$$2\sqrt{(2)}(\alpha+1)^{3/2}\int_0^\infty [(s^2-1)C(s) + 2sD(s)]K_s^{\,1}(\alpha)\,\mathrm{d}s = \frac{\alpha}{(\alpha^2-1)^{3/2}} + \tfrac{3}{2}(\alpha+1)\left(1 - \frac{\alpha}{\sqrt{(\alpha^2-1)}}\right).$$

Using (A1) and [73a], this can now be written as

$$2\sqrt{(2)}\int_0^\infty [(s^2-1)C(s) + 2sD(s) + sE(s)]K_s'(\alpha)\,\mathrm{d}s$$
$$= -\frac{1}{2(\alpha+1)^{3/2}(\alpha-1)} - \frac{3}{2(\alpha+1)^{3/2}} + \frac{3}{2(\alpha-1)^{1/2}(\alpha+1)}.$$

Then [A.6a], [A.9a] and [A.10a] imply that the inversion is

$$(s^2-1)C(s) + 2sD(s) + sE(s) = -\tfrac{1}{8}b(s) + \frac{3}{2\cosh s\pi} - \frac{3}{8(s^2+\frac{1}{4})}.$$

Substitution of [72b], [73b] and [A10b] then shows, after some manipulation, that

$$D(s) = \frac{\tanh s\pi}{8(s^2+\frac{1}{4})} + \frac{s}{\cosh s\pi}\left[\frac{1}{3} + \frac{1}{2(s^2+1)} - \frac{1}{4(s^2+\frac{1}{4})}\right] + \frac{s}{2(s^2+1)} - \frac{s}{2(s^2+\frac{1}{4})}. \qquad [74]$$

Now, since [31] implies that

$$\frac{1}{V}[q^{(1)}_{1z}]_{z=0} = w(\rho)\cos\phi \quad (\rho > 1) \tag{75}$$

where

$$w(\rho) = -\frac{1}{V\rho}\left(\frac{\partial^2\Psi_1^{(1)}}{\partial r\partial\theta} + X_1^{(1)}\right)_{\substack{r=\rho\\ \theta=\frac{1}{2}\pi}} \tag{76}$$

it follows from [17] that, in this approximation which ignores surface tension and gravity, the interface displacement function $f \equiv f(\rho)$ and satisfies

$$\left(1-\frac{1}{\rho}\right)^2\left(1+\frac{1}{2\rho}\right)\frac{\mathrm{d}f}{\mathrm{d}\rho} + \tfrac{3}{4}\left(1-\frac{1}{\rho^2}\right)\frac{f}{\rho^2} = w(\rho) \quad (\rho > 1). \tag{77}$$

Proceeding to evaluate $w(\rho)$, it follows from [47] that

$$\begin{aligned}
\frac{1}{V}\left(\frac{\partial\Psi_1^{(1)}}{\partial\theta}\right)_{\theta=\frac{1}{2}\pi} &= -\frac{1}{V}\sinh\xi\left(\frac{\partial\Psi_1^{(1)}}{\partial\eta}\right)_{\eta=0}\\
&= \sqrt{(2)}(\alpha+1)^{3/2}\int_0^\infty\left\{\frac{sC(s)}{\tanh\frac{1}{2}s\pi} + D(s)\left[\tanh\tfrac{1}{2}s\pi - \frac{2s^2}{\sinh s\pi}\right]\right\}K_s'(\alpha)\,\mathrm{d}s\\
&= \sqrt{(2)}(\alpha+1)^{3/2}\int_0^\infty\left\{\frac{1}{8(s^2+\frac{1}{4})} - \frac{1}{4\cosh s\pi} - \frac{1}{16(s^2+\frac{1}{4})\cosh s\pi}\right.\\
&\quad\left. - \frac{s}{\sinh s\pi}\left[\frac{2}{3} - \frac{1}{2(s^2+\frac{1}{4})} + \frac{1}{\cosh s\pi}\left(\frac{2}{3}s^2 + \frac{11}{6} - \frac{1}{8(s^2+\frac{1}{4})}\right)\right]\right\}K_s'(\alpha)\,\mathrm{d}s
\end{aligned}$$

after substitution of [72b] and [74] and some manipulation. Also from [48] and then [73b],

$$\begin{aligned}
\frac{1}{V}(X_1^{(1)})_{\theta=\frac{1}{2}\pi} &= \sqrt{(2)}(\alpha+1)^{1/2}\int_0^\infty E(s)\tanh\tfrac{1}{2}s\pi Ks^1(\alpha)\,\mathrm{d}s\\
&= \sqrt{(2)}(\alpha-1)^{1/2}(\alpha+1)\int_0^\infty\left[-\frac{(1-\operatorname{sech}s\pi)}{8(s^2+\frac{1}{4})} + \frac{s(1-\operatorname{sech}s\pi)}{\sinh s\pi}\left(\frac{1}{3} + \frac{1}{4(s^2+\frac{1}{4})}\right)\right]K_s'(\alpha)\,\mathrm{d}s.
\end{aligned}$$

These s-integrals can be evaluated or re-written by means of [A6a], [A9a], [A21], [A22], [A23], [A24a, b], [A25a, b] and [A26a, b] and the expressions thus obtained are

$$\begin{aligned}
\frac{1}{V}\left(\frac{\partial\Psi_1^{(1)}}{\partial\theta}\right)_{\theta=\frac{1}{2}\pi} &= \tfrac{1}{8}\left[\left(\frac{\alpha+1}{\alpha-1}\right)^{1/2} - 1\right]^2 - \frac{(\alpha+1)^{1/2}}{2\pi}\int_0^\infty\frac{(\cosh u - 1)\,\mathrm{d}u}{(\cosh u+\alpha)^{3/2}}\\
&\quad - \frac{(\alpha+1)^{3/2}}{8\pi(\alpha-1)}\int_0^\infty\frac{(\cosh u+1)}{(\cosh u+\alpha)^{3/2}}\,\mathrm{d}u\\
&\quad + \frac{(\alpha+1)^{3/2}}{2\pi}\int_0^\infty\left\{\cosh u + 1 + \left(\frac{\cosh u - 1}{\cosh u + 1}\right) + \tfrac{11}{4}(\cosh u - 1)\right\}\frac{\mathrm{d}u}{(\cosh u+\alpha)^{5/2}}\\
\frac{1}{V}(X_1^{(1)})_{\theta=\frac{1}{2}\pi} &= \tfrac{1}{4}\left[1 - \left(\frac{\alpha+1}{\alpha-1}\right)^{1/2}\right] + \frac{1}{2\pi(\alpha-1)^{1/2}}\int_0^\infty\frac{\mathrm{d}u}{(\cosh u+\alpha)^{1/2}}\\
&\quad - \frac{1}{2\pi}(\alpha+1)(\alpha-1)^{1/2}\int_0^\infty\frac{\mathrm{d}u}{(\cosh u+\alpha)^{5/2}}\\
&= -\tfrac{1}{4}\left[\left(\frac{\alpha+1}{\alpha-1}\right)^{1/2} - 1\right] + \frac{2}{3\pi(\alpha-1)^{1/2}}\int_0^\infty\frac{\cosh u\,\mathrm{d}u}{(\cosh u+\alpha)^{3/2}}
\end{aligned}$$

after an integration by parts. The substitution of [65] then yields

$$\frac{1}{V}\left(\frac{\partial \Psi_1^{(1)}}{\partial \theta}\right)_{\theta=\frac{1}{2}\pi} = \tfrac{1}{8}(\rho-1)^2 + \frac{\rho(\rho^2-1)}{4\pi}\int_0^1 (\rho^2-1+t)^{-3/2}\left[-2(1-t)-\tfrac{1}{2}\rho^2 + \frac{2\rho^2 t}{\rho^2-1+t} + \frac{\rho^2 t^2(1-t)}{\rho^2-1+t} + \frac{11\rho^2 t(1-t)}{2(\rho^2-1+t)}\right]\frac{dt}{\sqrt{(t(1-t))}}$$

$$\frac{1}{V}(X_1^{(1)})_{\theta=\frac{1}{2}\pi} = -\tfrac{1}{4}(\rho-1) + \frac{1}{6\pi}(\rho^2-1)^2\int_0^1 \frac{(2-t)}{(\rho^2-1+t)^{3/2}}\frac{dt}{\sqrt{(t(1-t))}}. \qquad [78]$$

The first of these expressions can be further simplified by writing

$$\frac{2\rho^2 t}{\rho^2-1+t} = 2t\left(1+\frac{1-t}{\rho^2-1+t}\right)$$

and using the following identities, obtained by integration by parts:

$$\int_0^1 \frac{t^{1/2}(1-t)^{1/2}}{(\rho^2-1+t)^{k+1/2}}dt = \frac{1}{2k-1}\int_0^1 \frac{(1-2t)}{(\rho^2-1+t)^{k-1/2}}\frac{dt}{\sqrt{(t(1-t))}} \qquad [79]$$

$$\int_0^1 \frac{t^{3/2}(1-t)^{1/2}}{(\rho^2-1+t)^{k+1/2}}dt = \frac{1}{2k-1}\int_0^1 \frac{(3-4t)}{(\rho^2-1+t)^{k-1/2}}\sqrt{\left(\frac{t}{1-t}\right)}dt$$

Thus, setting $k=2$ and then $k=1$,

$$\frac{1}{V}\left(\frac{\partial \Psi_1^{(1)}}{\partial \theta}\right)_{\theta=\frac{1}{2}\pi} = \tfrac{1}{8}(\rho-1)^2 + \rho\frac{(\rho^2-1)}{4\pi}\int_0^1 (\rho^2-1+t)^{-3/2}[-2(1-2t)-\tfrac{1}{2}\rho^2 + \tfrac{2}{3}(1-2t)+\rho^2 t(1-\tfrac{4}{3}t)+\tfrac{11}{6}\rho^2(1-2t)]\frac{dt}{\sqrt{(t(1-t))}}$$

$$= \tfrac{1}{8}(\rho-1)^2 - \frac{\rho}{3\pi}(\rho^2-1)^2\int_0^1\left(\frac{t}{\rho^2-1+t}\right)^{3/2}\frac{dt}{\sqrt{(1-t)}}. \qquad [80]$$

The dimensionless normal velocity amplitude $w(\rho)$, defined by [75], is now obtained, according to [76] by suitably combining [78] with the ρ-derivative of [80]. Thus

$$\rho w(\rho) = \frac{(\rho^2-1)^2}{3\pi}\int_0^1\left\{\left(\frac{t}{\rho^2-1+t}\right)^{3/2}\left[1+\frac{4\rho^2}{\rho^2-1}-\frac{3\rho^2}{\rho^2-1+t}\right]-\frac{1-\frac{1}{2}t}{t^{1/2}(\rho^2-1+t)^{3/2}}\right\}\frac{dt}{\sqrt{(1-t)}}.$$

But, from [79],

$$3\rho^2\int_0^1\frac{t^{3/2}\,dt}{(\rho^2-1+t)^{5/2}\sqrt{(1-t)}} = 3\int_0^1\left(\frac{t}{\rho^2-1+t}\right)^{3/2}\frac{dt}{\sqrt{(1-t)}}+\int_0^1\frac{(3-4t)}{(\rho^2-1+t)^{3/2}}\sqrt{\left(\frac{t}{1-t}\right)}dt$$

and hence

$$\rho w(\rho) = \frac{\rho^2-1}{\pi}\int_0^1(\rho^2-1+t)^{-3/2}\left\{-\tfrac{1}{3}[(\rho^2-1)(1-t)^2-\rho^2t^2]+t^2(2-\tfrac{3}{2}t) + \tfrac{1}{2}t[3(1-t)^2-\rho^2(3-4t)]\right\}\frac{dt}{\sqrt{(t(1-t))}}$$

$$= \frac{\rho^2-1}{\pi}\int_0^1\left(\frac{t}{\rho^2-1+t}\right)^{3/2}(2-\tfrac{3}{2}t)\frac{dt}{\sqrt{(1-t)}}$$

after further use of [79]. Thus

$$w(\rho)=\frac{\rho^2-1}{2\pi\rho}\int_0^1\left(\frac{t}{\rho^2-1+t}\right)^{3/2}\left[\frac{1}{\sqrt{(1-t)}}+3\sqrt{(1-t)}\right]\mathrm{d}t. \qquad [81]$$

Evidently, as $\rho\to\infty$,

$$w(\rho)\sim\frac{1}{2\pi\rho^2}[B(\tfrac{5}{2},\tfrac{1}{2})+3B(\tfrac{5}{2},\tfrac{3}{2})]+0\left(\frac{1}{\rho^4}\right) \qquad [82]$$

$$=\frac{9}{32\rho^2}+0\left(\frac{1}{\rho^4}\right)$$

whilst the asymptotic form [68] shows that, as $\rho\to1$,

$$w(\rho)\sim\frac{\rho^2-1}{2\pi\rho}\left[4-6(\rho^2-1)\log_e\left(\frac{1}{\rho^2-1}\right)+0(\rho^2-1)\right]$$

$$\sim\frac{4}{\pi}(\rho-1)-\frac{12}{\pi}(\rho-1)^2\log_e\left(\frac{1}{\rho-1}\right)+0[(\rho-1)^2]. \qquad [83]$$

Together with the differential equation [77], [83] suggests that the behaviour of $f(\rho)$ near $\rho=1$ is

$$f(\rho)\sim f_0+f_1(\rho-1)\log_e\left(\frac{1}{\rho-1}\right)+0(\rho-1)$$

$$=\frac{8}{3\pi}-\frac{4}{\pi}(\rho-1)\log_e\left(\frac{1}{\rho-1}\right)+0(\rho-1) \qquad [84]$$

with the coefficients determined by direct substitution. This is of the same structure as the expression [69] for the axisymmetric case (i). However, in contrast to that case, the constant f_0 can be determined by seeking the value of θ for which the $\hat{\theta}$ component of tangential stress at the surface of the sphere is zero. This is the appropriate condition for the attachment of the interface to the sphere, and to order λ, yields

$$0=\left[\frac{\partial}{\partial r}(q_{0\theta}+\lambda q_{1\theta}^{(1)})\right]_{r=1}$$

$$=[\tfrac{3}{2}\cos\theta-\lambda w'(1)]V\cos\phi,$$

from [75] and the alternative form of [7]. But $\cos\theta=\lambda f(1)$ here and so, with $w'(1)$ given by [83], it follows that $f(1)=8/3\pi$ as in [84].

The value of $f(\infty)$ depends on global features and cannot be deduced from a local analysis of the differential equation [77], using [82]. If the natural assumption that $f\to0$ as $\rho\to\infty$, is made, then [77] and [82] imply that

$$\frac{\mathrm{d}f}{\mathrm{d}\rho}\sim\frac{9}{32\rho^2}\quad\text{as }\rho\to\infty.$$

Also, since the integrand of [81] is uniformly positive, [77] has a positive right hand side $w(\rho)$. Thus as ρ decreases from infinity, f decreases from zero and [77] implies that $f'(\rho)$ must remain positive. Hence $f(\rho)$ is then an increasing function of ρ for all $\rho>1$ and cannot take the value

$8/3\pi$ at $\rho=1$ prescribed by [84]. The exact solution of [77] that vanishes at infinity is

$$f(\rho) = -\left(1-\frac{1}{\rho}\right)^{-1}\left(1+\frac{1}{2\rho}\right)^{-1/2}\int_\rho^\infty \frac{w(\rho')\,d\rho'}{\left(1-\frac{1}{\rho'}\right)\left(1+\frac{1}{2\rho'}\right)^{1/2}}. \quad [85]$$

This incompatibility shows that the assumed form of solution is invalid in the limit $\gamma = 0 = \Delta g$, in which the absence of surface tension and gravity forces means there is no "restoring" mechanism to act against the "deforming" mechanism created by the viscosity discontinuity. However it may be expected that for small enough values of γ and Δg, not both zero, the above calculation provides a good approximation to the flow field in regions where surface tension and gravity forces have little influence.

7. ACCURACY OF THE SOLUTIONS PRESENTED

On defining K (>0) by

$$K^2 = (\gamma/\mu V)^2 + (\Delta g/\mu V)^2 \quad [86]$$

the solutions given in section 5 for f and $V^{-1}\,\mathbf{q}_1^{(j)}$ are approximations of order $\mu V/\gamma$ and $\mu V/\Delta g$ to the $0(K^{-1})$ terms which would be obtained if [50] and [51] could be retained in the calculation. These and all higher order terms in the expansion of f and $V^{-1}\,\mathbf{q}_1^{(j)}$ in inverse powers of K are evidently functions of the parameter $\Delta g/\gamma$. Successive corrections to f are evaluated, according to [13], from the pressure discontinuity arising at the previous stage of the calculation. Thus in case (i), the pressure discontinuity in the given solution leads to axisymmetric corrections to f and $V^{-1}\mathbf{q}_1^{(j)}$ and thus a force of order $(\mu_1-\mu_2)V/K^2$ on the sphere in the $\hat{z}$ direction. Evidently all subsequent corrections are also axisymmetric and the expression in curly brackets in [54] becomes a power series in K^{-1} with coefficients depending on $\Delta g/\gamma$. Meanwhile, in the asymmetric case (ii), the pressure discontinuity in the given solution has meridianal variation of the form 1 and cos 2ϕ, which leads to terms of order K^{-2} in $f(\rho,\phi)$ with similar dependence on ϕ. Hence, from [12], the terms of order K^{-2} in $q_{1z}^{(j)}$ are proportional to cos ϕ and cos 3ϕ and the first type yield a torque of order $(\mu_1-\mu_2)V/K^2$ on the sphere in the $\hat{y}$ direction. Similarly the terms of order K^{-3} in f and $\mathbf{q}_1^{(j)}$ produce a force of order $(\mu_1-\mu_2)V/K^3$ on the sphere in the $\hat{x}$ direction. Evidently the subsequent corrections modify the torque and force alternately and the expressions in [58] and [59] are modified by the suitable insertion of power series in K^{-2} with coefficients depending on $\Delta g/\gamma$. With this procedure established, the exact solution for f and $\mathbf{q}_1^{(j)}$ may subsequently be regarded as known.

If the perturbation scheme introduced by [4] and [5] is continued to second order in λ, then, since [8] needs no modification, the equations corresponding to [9] and [10] are

$$\hat{n} = \hat{z}\left\{1-\tfrac{1}{2}\lambda^2\left[\left(\frac{\partial f}{\partial\rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial\phi}\right)^2\right]\right\} - \lambda\frac{\partial f}{\partial\rho}\hat{\rho} - \frac{\lambda}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi} + 0(\lambda^2) \quad [87]$$

$$\mathbf{q}(\rho,\phi,\lambda f(\rho,\phi)) = \mathbf{q}_0(\rho,\phi,0) + \lambda\mathbf{q}_1^{(j)}(\rho,\phi,0) + \lambda^2\mathbf{q}_2^{(j)}((\rho,\phi,0)$$

$$+\left[\frac{\partial\mathbf{q}_0}{\partial z}(\rho,\phi,0)+\lambda\frac{\partial\mathbf{q}_1^{(j)}}{\partial z}(\rho,\phi,0)\right]\lambda f(\rho,\phi) + \tfrac{1}{2}\frac{\partial^2\mathbf{q}_0}{\partial z^2}(\rho,\phi,0)\lambda^2[f(\rho,\phi)]^2$$

$$+0(\lambda^3) \quad (j=1,2). \quad [88]$$

The continuity of $\mathbf{q}_0$ and its derivatives, $\mathbf{q}_1^{(j)}$ and $\partial\mathbf{q}_1^{(j)}/\partial z$ at $z=0$ implies, in [88], that $\mathbf{q}_2^{(j)}$ is also continuous at $z=0$. Now [19] and the continuity of $\partial q_{1z}^{(j)}/\partial z$ imply that $[\partial q_{1z}^{(j)}/\partial z]_{z=0}=0$. Hence, using [21] and [87], the $0(\lambda^2)$ terms in the equation $\mathbf{q}.\hat{n}=0$ show that $[q_{2z}^{(j)}]_{z=0}=0$. Also,

[14] and $(\partial/\partial z)\text{div } \mathbf{q}_1^{(j)} = 0$ imply that $\partial^2 q_{1z}^{(j)}/\partial z^2$ is continuous at $z = 0$, [19] shows that $\partial P_1^{(j)}/\partial z$ is continuous whilst div $\mathbf{q}_2^{(j)} = 0$ and the continuity of $\mathbf{q}_2$ imply that $\partial q_{2z}^{(j)}/\partial z$ is also continuous at $z = 0$. Hence

$$\frac{1}{\mu}[\sigma_{zz}^{(1)} - \sigma_{zz}^{(2)}]_{z=\lambda f} = 2\lambda\left(-P_0 + 2\frac{\partial q_{0z}}{\partial z}\right)_{z=0} - \lambda(P_1^{(1)} - P_1^{(2)})_{z=0} - \lambda^2(P_2^{(1)} - P_2^{(2)})_{z=0}.$$

Equations [14], [20] and the vanishing of $q_{2z}^{(j)}$ and $\partial q_{1z}^{(j)}/\partial z$ at $z = 0$ show that

$$\frac{1}{\mu}[\sigma_{\rho z}^{(1)} - \sigma_{\rho z}^{(2)}]_{z=\lambda f} = 2\lambda^2\left[\frac{\partial q_{1\rho}^{(1)}}{\partial z} + \frac{\partial q_{1z}^{(1)}}{\partial \rho} + f\left(\frac{\partial^2 q_{0\rho}}{\partial z^2} + \frac{\partial^2 q_{0z}}{\partial\rho\partial z} + \frac{\partial^2 q_{1\rho}^{(1)}}{\partial z^2}\right)\right]_{z=0} + \lambda^2\left[\frac{\partial q_{2\rho}^{(1)}}{\partial z} - \frac{\partial q_{2\rho}^{(2)}}{\partial z}\right]_{z=0}$$

$$\frac{1}{\mu}[\sigma_{\phi z}^{(1)} - \sigma_{\phi z}^{(2)}]_{z=\lambda f} = 2\lambda^2\left[\frac{\partial q_{1\phi}^{(1)}}{\partial_z} + \frac{1}{\rho}\frac{\partial q_{1z}^{(1)}}{\partial \phi} + f\left(\frac{\partial^2 q_{0\phi}}{\partial z^2} + \frac{1}{\rho}\frac{\partial^2 q_{0z}}{\partial\phi\partial z} + \frac{\partial^2 q_{1\phi}^{(1)}}{\partial z^2}\right)\right]_{z=0} + \lambda^2\left[\frac{\partial q_{2\phi}^{(1)}}{\partial z} - \frac{\partial q_{2\phi}^{(2)}}{\partial z}\right]_{z=0}.$$

Hence the continuity at the interface of the stress components of order λ^2 requires that $P_2^{(j)}$ be continuous whilst $\partial q_{2\rho}^{(j)}/\partial z$ and $\partial q_{2\phi}^{(j)}/\partial z$ have discontinuities at $z = 0$ prescribed by the lower order velocity fields. Thus, in exact contrast to [19] and [20],

$$\left.\begin{aligned} q_{2\rho}^{(2)}(\rho,\phi,-z) &= q_{2\rho}^{(1)}(\rho,\phi,z), \quad q_{2\phi}^{(2)}(\rho,\phi,-z) = q_{2\phi}^{(1)}(\rho,\phi,z) \\ P_2^{(2)}(\rho,\phi,-z) &= P_2^{(1)}(\rho,\phi,z), \quad q_{2z}^{(2)}(\rho,\phi,-z) = -q_{2z}^{(1)}(\rho,\phi,z) \end{aligned}\right\} (z>0, r>1).$$

So in the axisymmetric case (i), the stream function is an odd function of z and there is zero contribution of relative order λ^2 to the force expression [54]. Similarly in the asymmetric case (ii), there are no contributions of relative order λ^2 to the force and torque expressions [58] and [59].

By contrast, the calculations in section 6 for zero γ and Δg have not yielded uniform approximations to f and $V^{-1}\mathbf{q}_1^{(j)}$ at small K. The solution presented in the axisymmetric case (i) is a good approximation for $\gamma = 0$ and $\Delta g/\mu V$ small, with the same reservations concerning the infinite derivative of f as $\rho \to 1$. But if $\gamma \neq 0$, the neglected surface tension terms in [16], which are of relative order $(\gamma/\mu V)(\rho - 1)^{-1}$, become unbounded as $\rho \to 1$ and the validity of the constructed solution must be confined to an outer region where surface tension is unimportant. The flow field pattern must be completed by constructing an inner solution valid near $z = 0$, $\rho = 1$ and extending to $\rho = 1 + 0(\gamma/\mu V)$. In case (ii), where the constructed outer solution is incompatible with a finite f at $\rho = 1$, the neglected surface tension and gravity terms in [18] are, according to [85], of relative order $(\gamma/\mu V)(\rho - 1)^{-3}$ and $(\Delta g/\mu V)(\rho - 1)^{-1}$ respectively. Thus the required inner region must extend to values of $(\rho - 1)$ of order $(\gamma/\mu V)^{1/3}$ or $(\Delta g/\mu V)$, whichever is the larger.

For large or small values of K, defined by [86], it has been shown that, though both surface tension and gravity forces act to stabilize the interface against the effects of the viscous stresses, gravity tends to confine the displacement near the sphere whilst surface tension tends to maintain the flatness of the interface. Further, a sufficiently large value of K is required to validate Ranger's suggestion that an axisymmetric Stokes flow past a sphere can provide a useful approximation to the corresponding two phase flow in which the sphere straddles the interface.

REFERENCES

ABRAMOVITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions.* Dover, New York.

BRENNER, H. & DAVIS, A. M. J. 1981 Emulsions containing a third solid internal phase. *J. Engng Mech. Div. ASCE* **107**, 609–621.

DAVIS, A. M. J., O'NEILL, M. E. & RANGER, K. B. 1975 Some properties of multi-fluid Stokes flows. *Int. J. Multiphase Flow* **2**, 1–8.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1965 *Tables of Integrals, Series and Products.* Academic Press, New York.

MICHAEL, D. H. 1964 Note on plane Stokes flow of adjoining viscous fluids at a corner. *Appl. Sci. Res.* **A13**, 349–352.

PAYNE, L. E. & PELL, W. H. 1960 The Stokes flow problem of a class of axially symmetric bodies. *J. Fluid Mech.* **7**, 529–549.

RANGER, K. B. 1978 The circular disk straddling the interface of a two-phase flow. *Int. J. Multiphase Flow* **4**, 263–277.

SCHNEIDER, J. C., O'NEILL, M. E. & BRENNER, H. 1973 On the slow viscous rotation of a body straddling the interface between two immiscible semi-infinite fluids. *Mathematika* **20**, 175–196.

SNEDDON, I. N., 1972 *The Use of Integral Transforms.* McGraw–Hill, New York.

Handbook of Chemistry and Physics. 1979 59th Edn. Ed. R. C. Weast. C.R.C. Press, New York.

APPENDIX

The Mehler conal function $K_s{}^m(\alpha)$ satisfies the differential equation [42], namely

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha^2-1)\frac{\mathrm{d}K_s{}^m}{\mathrm{d}\alpha}\right]+\left(s^2+\tfrac{1}{4}-\frac{m^2}{\alpha^2-1}\right)K_s{}^m=0 \quad (\alpha>1)$$

where m is a non-negative integer and $s \geq 0$. Denoting by $K_s{}^{(m)}$ the mth derivative of $K_s(\alpha)$, the functions of positive order m are given in terms of the zero order conal function by

$$K_s{}^m(\alpha)=(\alpha^2-1)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\alpha^m}K_s=(\alpha^2-1)^{m/2}K_s{}^{(m)}(\alpha) \qquad [A1]$$

Substitution of [A1] in [42] shows that successive derivatives of $K_s(\alpha)$ are related by the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\{(\alpha^2-1)^m K_s{}^{(m)}\}+[s^2+(m-\tfrac{1}{2})^2](\alpha^2-1)^{m-1}K_s{}^{(m-1)}=0 \quad (m\geq 1).$$

Now, the Fock theorem states that if

$$F(\alpha)=\int_0^\infty G(s)K_s(\alpha)\,\mathrm{d}s \quad (\alpha\geq 1),$$

then

$$G(s)=s\tanh s\pi\int_0^\infty F(\alpha)K_s(\alpha)\,\mathrm{d}\alpha \quad (s\geq 0). \qquad [A2]$$

The above recurrence relation enables the corresponding result for $K_s{}^{(m)}$ to be deduced inductively. Thus, for each $m \geq 1$, if

$$F_m(\alpha)=\int_0^\infty G_m(s)K_s{}^{(m)}(\alpha)\,\mathrm{d}s \quad (\alpha\geq 1)$$

then this Mehler–Fock transform of order m has the inversion formula

$$G_m(s) = \frac{s \tanh s\pi}{\prod_{r=1}^{m} [s^2 + (r - \frac{1}{2})^2]} \int_1^\infty (\alpha^2 - 1)^m F_m(\alpha) K_s^{(m)}(\alpha)\, d\alpha \quad (s \geq 0). \qquad [A3]$$

For further details, see Sneddon (1972), section 7.

As given by Schneider, O'Neill & Brenner (1973), the identity

$$(\alpha - \cos \eta)^{-1/2} = \sqrt{(2)} \int_0^\infty \frac{\cosh s(\pi - \eta)}{\cosh s\pi} K_s(\alpha)\, ds \qquad [A4a]$$

implies, by [A2]

$$\int_1^\infty \frac{K_s(\alpha)}{(\alpha - \cos \eta)^{1/2}}\, d\alpha = \sqrt{(2)} \frac{\cosh s(\pi - \eta)}{s \sinh s\pi} \qquad [A4b]$$

whilst

$$\sin \eta\, (\alpha - \cos \eta)^{-3/2} = 2\sqrt{(2)} \int_0^\infty \frac{\sinh s(\pi - \eta)}{\cosh s\pi} K_s(\alpha)\, ds \qquad [A5a]$$

implies

$$\int_1^\infty \frac{K_s(\alpha)}{(\alpha - \cos \eta)^{3/2}}\, d\alpha = \frac{2\sqrt{(2)} \sinh s(\pi - \eta)}{\sin \eta \sinh s\pi} \qquad [A5b]$$

Also, formula (A3) shows that

$$(\alpha - \cos \eta)^{-3/2} = -2\sqrt{(2)} \int_0^\infty \frac{\cosh s(\pi - \eta)}{\cosh s\pi} K_s'(\alpha)\, ds \qquad [A6a]$$

implies

$$\int_1^\infty \frac{\alpha^2 - 1}{(\alpha - \cos \eta)^{3/2}} K_s'(\alpha)\, d\alpha = \frac{-2\sqrt{(2)}\,(s^2 + \frac{1}{4}) \cosh s(\pi - \eta)}{s \sinh s\pi}. \qquad [A6b]$$

An η-integral of $\sin \eta$ times [A6a] is

$$(\alpha - 1)^{-1/2} - (\alpha + 1)^{-1/2} = -\sqrt{(2)} \int_0^\infty \frac{\cosh s\pi + 1}{(s^2 + 1) \cosh s\pi} K_s'(\alpha)\, ds. \qquad [A7]$$

The function $a(s)$ defined by the equation

$$\int_0^\infty a(s) K_s'(\alpha)\, ds = \frac{1}{2\sqrt{(2)}} \left[-\frac{\alpha - 1}{(\alpha + 1)^{3/2}} + \frac{(\alpha - 1)^{1/2}}{\alpha + 1} \right] \qquad [A8a]$$

$$= \frac{1}{2\sqrt{(2)}} \left[\frac{2}{(\alpha + 1)^{3/2}} + \frac{1}{(\alpha - 1)^{1/2}} - \frac{1}{(\alpha + 1)^{1/2}} - \frac{2}{(\alpha + 1)(\alpha - 1)^{1/2}} \right]$$

is given, according to [A3], [A6a] and [A7], by

$$a(s) = -\frac{2}{\cosh s\pi} - \frac{\cosh s\pi + 1}{2(s^2+1)\cosh s\pi} - \frac{s \tanh s\pi}{\surd(2)(s^2+\frac{1}{4})}\int_1^\infty (\alpha-1)^{1/2} K_s'(\alpha)\,d\alpha$$

$$= -\frac{2}{\cosh s\pi} - \frac{\cosh s\pi + 1}{2(s^2+1)\cosh s\pi} + \frac{1}{2(s^2+\frac{1}{4})}$$

because, using [A4b],

$$\frac{1}{\surd(2)}\int_1^\infty (\alpha-1)^{1/2} K_s'(\alpha)\,d\alpha = -\frac{1}{2\surd(2)}\int_1^\infty \frac{K_s(\alpha)}{(\alpha-1)^{1/2}}\,d\alpha = \frac{-1}{2s\tanh s\pi} \qquad \text{[A9b]}$$

The inverses of [A8a] and [A9b] are

$$\frac{1}{2\surd(2)}\int_1^\infty (\alpha-1)^{3/2}\left[1-\left(\frac{\alpha-1}{\alpha+1}\right)^{1/2}\right] K_s'(\alpha)\,d\alpha = \frac{(s^2+\frac{1}{4})a(s)}{s\tanh s\pi}$$

$$= -\frac{2(s^2+\frac{1}{4})}{s\sinh s\pi} - \frac{(s^2+\frac{1}{4})(\cosh s\pi+1)}{2(s^2+1)s\sinh s\pi} + \frac{1}{2s\tanh s\pi} \qquad \text{[A8b]}$$

$$\frac{1}{(\alpha+1)(\alpha-1)^{1/2}} = -\frac{1}{\surd(2)}\int_0^\infty \frac{K_s'(\alpha)\,ds}{s^2+\frac{1}{4}} \qquad \text{[A9a]}$$

It is often possible to reduce by one the order of a $K_s^{(m)}$ inversion by considering $(d/d\alpha)\{(\alpha^2-1)^m F_m(\alpha)\}$, but this procedure fails when

$$-\int_1^\infty \frac{d}{d\alpha}[(\alpha^2-1)^m F_m(\alpha)] K_s^{(m-1)}(\alpha)\,d\alpha \neq \int_1^\infty (\alpha^2-1)^m F_m(\alpha) K_s^{(m)}(\alpha)\,d\alpha.$$

This occurs when $(\alpha^2-1)^m F_m(\alpha)$ tends to a non-zero limit as $\alpha \to 1$, the reason being that in this case the additional factor $(s^2+(m-\frac{1}{2})^2)$ makes the s-integral divergent. For example, the correct inversion of

$$\int_0^\infty b(s) K_s'(\alpha)\,ds = \frac{\surd(2)}{(\alpha+1)^{3/2}(\alpha-1)} \qquad \text{[A10a]}$$

is

$$b(s)\frac{s^2+\frac{1}{4}}{s\tanh s\pi} = \surd(2)\int_1^\infty K_s'(\alpha)\frac{d\alpha}{(\alpha+1)^{1/2}} = -1 + \frac{2s}{\sinh s\pi},$$

by use of [4.22] and [A5b]. Thus

$$b(s) = -\frac{s\tanh s\pi}{s^2+\frac{1}{4}} + \frac{2s^2}{(s^2+\frac{1}{4})\cosh s\pi} \qquad \text{(A10b)}$$

and the alternative procedure fails here because

$$(s^2+\tfrac{1}{4})b(s) \sim -s \quad \text{as} \quad s \to \infty.$$

APPENDIX 2

Consider the integral

$$J_{n,m}(s) = \frac{s\tanh s\pi}{\surd(2)}\int_1^\infty \frac{(\alpha-1)^{n+m-1/2}}{(\alpha+1)^{n+1}} K_s^{(m)}(\alpha)\,d\alpha \ (n \geq 0, m \geq 0) \qquad \text{[A11]}$$

which, according to [A3], is such that

$$\frac{1}{\sqrt{(2)}}\frac{(\alpha-1)^{n-1/2}}{(\alpha+1)^{n+m+1}}=\int_0^\infty \frac{J_{n,m}(s)K_s^{(m)}(\alpha)}{\prod_{r=1}^{m}[s^2+(r-\frac{1}{2})^2]}\,\mathrm{d}s. \qquad \text{[A12]}$$

On substituting the integral representation

$$K_s(\alpha)=\frac{\sqrt{(2)}}{\pi}\cosh s\pi\int_0^\infty \frac{\cos su}{(\cosh u+\alpha)^{1/2}}\,\mathrm{d}u \qquad \text{[A13]}$$

(Abramovitz & Stegun, 1964) into [A11] and then writing

$$\alpha-1=(\cosh u+1)\operatorname{cosech}^2 v,$$

it follows that

$$\begin{aligned} J_{n,m}(s)&=\frac{2}{\pi}s\sinh s\pi(-1)^m\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^\infty\int_0^\infty\frac{\cos su\,(\cosh u+1)^n\sinh v\,\mathrm{d}u\,\mathrm{d}v}{(\cosh u+\cosh 2v)^{n+1}\cosh^{2m}v}\\ &=\frac{2}{\pi}s\sinh s\pi\,(-1)^m\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^\infty\int_0^\infty\frac{\cos su\,\mathrm{d}u}{\cosh^{2m}v}\sum_{r=0}^{n}{}^nC_r\frac{(-2)^r\sinh^{2r+1}v\,\mathrm{d}v}{(\cosh u+\cosh 2v)^{r+1}}\\ &=\frac{s}{2\pi}\sinh s\pi(-1)^m\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_1^\infty\int_0^\infty\cos su\,\mathrm{d}u\left(\frac{2}{V+1}\right)^{m+1/2}\sum_{r=0}^{n}{}^nC_r\frac{(-1)^r(V-1)^r\,\mathrm{d}V}{(\cosh u+V)^{r+1}}\end{aligned}$$

where $V=\cosh 2v$. The u-integrals are now derivatives with respect to V of the function

$$g(v,s)=\frac{s}{\pi}\sinh s\pi\int_0^\infty\frac{\cos su\,\mathrm{d}u}{(\cosh u+\cosh 2v)}=\frac{s\sin 2vs}{\sinh 2v} \qquad \text{[A14]}$$

(Gradshteyn & Ryzhik, 1965, section 3.983) and subsequent use of Leibnitz theorem and integration by parts yields

$$\begin{aligned} J_{n,m}(s)&=\tfrac{1}{2}(-1)^m\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_1^\infty\left(\frac{2}{V+1}\right)^{m+1/2}\sum_{r=0}^{n}{}^nC_r\frac{(V-1)^r}{r!}\frac{\partial^r g}{\partial V^r}g\,\mathrm{d}V\\ &=\frac{1}{2}\frac{(-1)^m}{n!}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_1^\infty\left(\frac{2}{V+1}\right)^{m+\frac{1}{2}}\frac{\partial^n}{\partial V^n}[(V-D_g{}^n]\,\mathrm{d}V\\ &=\frac{1}{2}\frac{(-1)^m}{n!}\frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_1^\infty\left(\frac{2}{V+1}\right)^{m+1/2}\left(\frac{V-1}{V+1}\right)^n g\,\mathrm{d}V\\ &=\frac{(-1)^m}{n!}\frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^\infty\frac{s\sin 2vs}{\cosh^{2m+1}v}\tanh^{2n}v\,\mathrm{d}v. \qquad \text{[A15]}\end{aligned}$$

Evidently [A15] satisfies the recurrence relation

$$J_{n,m}=-(n+m-\tfrac{1}{2})J_{n,m-1}+(n+1)J_{n+1,m-1}\quad(m\geqslant 1,n\geqslant 0)$$

which can be deduced from [A11] by means of the differential equation [42].

APPENDIX 3

Some standard integrals, given by Gradshteyn & Ryzhik (1965) are

$$\int_0^\infty \frac{ds}{\cosh sq} = \frac{\pi}{2q} \quad (q>0) \qquad [A16]$$

$$\int_0^\infty \frac{\cosh s\pi + 1}{(s^2+1)\cosh s\pi}\, ds = 2, \int_0^\infty \frac{s \sin 2vs}{s^2+\frac{1}{4}}\, ds = \tfrac{1}{2}\pi\, e^{-v} \quad (v>0) \qquad [A17]$$

$$\int_0^\infty \frac{s \cos su}{\sinh s\pi} \cosh s(\pi-\eta)\, ds = \tfrac{1}{2}\frac{d}{du}\left(\frac{\sinh u}{\cosh u - \cos \eta}\right) \quad (0<\eta<2\pi). \qquad [A18]$$

A particular case of [A18] is

$$\int_0^\infty \frac{s \cos su}{\sinh s\pi}\, ds = \frac{1}{2(\cosh u + 1)} \qquad [A19]$$

which, when expanded in powers of u, yields

$$\int_0^\infty \frac{s\, ds}{\sinh s\pi} = \tfrac{1}{4}, \quad \int_0^\infty \frac{s^3\, ds}{\text{siih}\, s\pi} = \tfrac{1}{8}, \quad \int_0^\infty \frac{s^5\, ds}{\sinh s\pi} = \tfrac{1}{4}. \qquad [A20]$$

Integrals involving $K_s'(\alpha)$ can often be rewritten by means of the integral representation [A13]. Thus, using [A19],

$$-\surd(2)\int_0^\infty \frac{sK_s'(\alpha)\, ds}{\sinh s\pi \cosh s\pi} = \frac{1}{2\pi}\int_0^\infty \frac{du}{(\cosh u + 1)(\cosh u + \alpha)^{3/2}}$$

$$= \frac{3}{4\pi}\int_0^\infty \frac{(\cosh u - 1)\, du}{(\cosh u + \alpha)^{5/2}} \qquad [A21]$$

by integration by parts and

$$-2\surd(2)\int_0^\infty \frac{s(2s^2+1)K_s'(\alpha)\, ds}{\sinh s\pi \cosh s\pi} = \frac{1}{\pi}\int_0^\infty \frac{du}{(\cosh u + \alpha)^{3/2}}\left[\left(1 - 2\frac{d^2}{du^2}\right)\frac{1}{\cosh u + 1}\right]$$

$$= \frac{3}{2\pi}\int_0^\infty \frac{(\cosh u - 1)\, du}{(\cosh u + \alpha)^{5/2}} + \frac{3}{\pi}\int_0^\infty \left(\frac{\cosh u - 1}{\cosh u + 1}\right)\frac{du}{(\cosh u + \alpha)^{5/2}}. \qquad [A22]$$

Also, using [A18]

$$-\surd(2)\int_0^\infty \frac{sK_s'(\alpha)\, ds}{\sinh s\pi} = -\surd(2)\lim_{\eta\to 0}\int_0^\infty \frac{sK_s'(\alpha)\cosh s(\pi-\eta)}{\sinh s\pi \cosh s\pi}\, ds$$

$$= \frac{1}{2\pi}\lim_{\eta\to 0}\int_0^\infty \frac{du}{(\cosh u + \alpha)^{3/2}}\frac{d}{du}\left(\frac{\sinh u}{\cosh u - \cos \eta}\right)$$

$$= \frac{3}{4\pi}\int_0^\infty \frac{(\cosh u + 1)\, du}{(\cosh u + \alpha)^{5/2}} \qquad [A23]$$

If the integrand differs only by a factor $(s^2+\frac{1}{4})^{-1}$ from a previously considered s-integral, the differential equation

$$\frac{d^2}{d\alpha^2}[(\alpha^2-1)K_s'(\alpha)] + (s^2+\tfrac{1}{4})\, K_s'(\alpha) = 0,$$

derived from [42], provides a simpler alternative method of evaluation. Thus if

$$I_1(\alpha) = -\sqrt{(2)}\int_0^\infty \frac{K_s'(\alpha)\,\mathrm{d}s}{(s^2+\frac{1}{4})\cosh s\pi}, \qquad \text{[A24a]}$$

then

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}[(\alpha^2-1)I_1] = \sqrt{(2)}\int_0^\infty \frac{K_s'(\alpha)\,\mathrm{d}s}{\cosh s\pi} = -\tfrac{1}{2}(\alpha+1)^{-3/2}$$

by [A6a]. But [45] and [A16] imply that $I_1(1) = 1/2\sqrt{2}$ and hence

$$I_1(\alpha) = \frac{2}{(\alpha-1)(\alpha+1)^{1/2}} - \frac{2\sqrt{(2)}}{\alpha^2-1} \qquad \text{[A24b]}$$

Next, if

$$I_2(\alpha) = -\sqrt{(2)}\int_0^\infty \frac{sK_s'(\alpha)\,\mathrm{d}s}{(s^2+\frac{1}{4})\sinh s\pi} \qquad \text{[A25a]}$$

then

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}[(\alpha^2-1)I_2] = \sqrt{(2)}\int_0^\infty \frac{sK_s'(\alpha)\,\mathrm{d}s}{\sinh s\pi} = -\frac{3}{4\pi}\int_0^\infty \frac{(\cosh u+1)}{(\cosh u+\alpha)^{5/2}}\mathrm{d}u$$
$$= \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left\{\frac{\alpha-1}{\pi}\int_0^\infty \frac{(\cosh u-1)}{(\cosh u+\alpha)^{3/2}}\mathrm{d}u\right\}$$

by use of first [A23] and then the identity

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left\{\frac{(\alpha-1)(\cosh u-1)}{(\cosh u+\alpha)^{3/2}} + \frac{3(\cosh u+1)}{4(\cosh u+\alpha)^{5/2}}\right\} = \frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{\sinh u}{(\cosh u+\alpha)^{5/2}}\right].$$

But [45] and [A20] imply that $I_2(1) = 1/4\sqrt{(2)}$ and hence

$$I_2(\alpha) = \frac{1}{\pi(\alpha+1)}\int_0^\infty \frac{(\cosh u-1)}{(\cosh u+\alpha)^{3/2}}\mathrm{d}u \qquad \text{[A25b]}$$

after using [A16] to show that

$$\int_0^\infty \frac{\cosh u-1}{(\cosh u+1)^{3/2}}\mathrm{d}u = \frac{1}{2\sqrt{(2)}}\int_0^\infty \frac{\mathrm{d}u}{\cosh\frac{1}{2}u} = \frac{\pi}{2\sqrt{(2)}}. \qquad \text{[A26]}$$

Lastly, if

$$I_3(\alpha) = -\sqrt{(2)}\int_0^\infty \frac{sK_s'(\alpha)\,\mathrm{d}s}{(s^2+\frac{1}{4})\sinh s\pi\cosh s\pi} \qquad \text{[A27a]}$$

then

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}[(\alpha^2-1)I_3] = \sqrt{(2)}\int_0^\infty \frac{sK_s'(\alpha)\,\mathrm{d}s}{\sinh s\pi\cosh s\pi} = -\frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}u}{(\cosh u+1)(\cosh u+\alpha)^{3/2}}$$

by [A21]. But [45] and [A20] imply that $I_3(1) = 1/8\surd 2$ and hence, using [A26] again,

$$(\alpha^2 - 1)I_3(\alpha) = \frac{2}{\pi}\int_0^\infty \frac{\cosh u + \alpha)^{1/2}}{\cosh u + 1}\,\mathrm{d}u - \surd(2)$$

After an integration by parts it follows that

$$I_3(\alpha) = \frac{1}{\pi(\alpha - 1)}\int_0^\infty \frac{(\cosh u + 1)\,\mathrm{d}u}{(\cosh u + \alpha)^{3/2}} - \frac{\surd(2)}{\alpha^2 - 1}. \qquad \text{[A27b]}$$